

INTRODUCTION TO COMBINATORICS AND GRAPH THEORY (Math-2031)



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Chapter 1

ELEMENTARY COUNTING PRINCIPLES

INTRODUCTION

This chapter develops some techniques for determining, without direct enumeration, the number of possible outcomes of a particular event or the number of elements in a set. Such sophisticated counting is sometimes called combinatorial analysis. It includes the study of permutations and combinations.

1.1 Basic Counting Principles

The Addition principle (The Sum Rule)

If a first task can be done in n_1 ways and a second task can be done in n_2 ways and if these tasks *can't be done at the same time*, then there are $n_1 + n_2$ ways to do either task. In other word, if $A_1, A_2, \dots, A_k$ are pair wise disjoint sets (i.e. $A_i \cap A_j = \emptyset$ where $i \neq j$), then the number of elements in their union is

$$|A_1 \cup A_2 \cup \dots \cup A_k| = |A_1| + |A_2| + \dots + |A_k| = \sum_{i=1}^k |A_i|$$

Remark 1 Let A be a finite set. The **cardinality** of the set A denoted by $|A|$ is the number of distinct elements contained in A .

Example 1.1.1 A student can choose a computer project from one of the three lists. The three lists contain 20, 14 and 11 possible projects, respectively. How many possible projects are there to choose from?

Solution

The student can choose a project from the first list in 20 ways, from the 2nd list in 14 ways and from the 3rd list in 11 ways.

Hence there are $20 + 14 + 11 = 45$ possible projects to choose from.

Example 1.1.2 When I was going to Addis, I met a man with 5 wives, 25 children, 52 grand children and 127 great-grand children. Assuming these sets do not overlap, how many people did I meet?

Exercise 1.1.1 The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors?

The Multiplication principle (The Product Rule)

Suppose that a procedure can be broken down in to two tasks, If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, Then there are $n_1 \times n_2$ ways to do the procedure.

Example 1.1.3 The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution

The procedure of labeling a chair consists of two tasks that is assigning one of the 26 Letters and then assigning one of the 100 possible integers to the seat. The product rule shows that there are $26 \times 100 = 2600$ different ways that a chair can be labeled. Therefore the largest number of chairs that can be labeled differently is 2600.

Exercise 1.1.2

1. A restaurant serves three types of starter, six main courses and five desserts. How many different two course meals (including a main course) are there?(Ans.48)
2. Abebe has 3 T-Shirts (red, white, yellow); 2 pairs of jeans (black, blue) and 2 pairs of shoes (takkies, sandals). How many different outfit combinations can he put together? (Ans.12)
3. Chris wants to buy a car. He can buy a Toyota, Nissan or Ford. He can choose a two-door, four-door or hatch-back model. The cars are available in red, white, silver or black. How many options can he choose from?(Ans.36)
4. How many different 9 digit cell phone number can be made if the first digit being odd, no digit may be a zero, and repetition is not allowed? (Ans. 201,600)
5. There are two roads between towns A and B. There are three roads between towns B and C, then how many different routes may one travel between towns A and C.
6. In a university, There are 15 exams that can be scheduled on Tuesday at 7pm and 5 (different from the first) exams that can be scheduled on Tuesday at 8pm. How many different exams are scheduled on Tuesday?

1.2 Inclusion-Exclusion Principle

When two tasks can be done at the same time, we can't use the addition principle to count the number of ways to do one of the two tasks. Adding the number of ways to do each task leads to an over count, since the ways to do both tasks are counted twice. To correctly count the number of ways to do one of the two tasks, we add the number of ways to do each of the two tasks and then subtract the number of ways to do both tasks. This technique is called the **principle of inclusion-exclusion**.

Theorem 1.2.1 Let A_1 and A_2 be two sets with cardinalities $|A_1|$ and $|A_2|$, then the principle of inclusion and exclusion states that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Note

Extending the above for three sets A_1, A_2 and A_3 by using distributive law. We obtain

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Example 1.2.1 In an advertising survey conducted on 200 people, it was found that 140 drink tea, 80 drink coffee and 40 drink both. Find how many drink at least one beverage and how many drinks neither?

Solution

Let A denote the set of tea drinkers, B the set of coffee drinkers. Then

$$|A \cup B| = |A| + |B| - |A \cap B| = 180$$

Hence 180 peoples drink at least one beverage and $200 - 180 = 20$ drink neither.

Note

For any real number x , the **ceiling** of x , written $\lceil x \rceil$, is the least integer greater than or equal to x and the **floor** of x , written by $\lfloor x \rfloor$, is the greatest integer less than or equal to x . Therefore, We use $\lfloor x \rfloor$ to denote the largest integer that is smaller than or equal to x . So the number of multiples of x that do not exceed n is $\lfloor n/x \rfloor$.

Example 1.2.2 How many integers are between 1 and 250 that are divisible by any of the integers 2, 3, 5 and 7.

Solution

$$|A_1| = \lfloor 250/2 \rfloor = 125, |A_2| = \lfloor \frac{250}{3} \rfloor = 83, |A_3| = \lfloor \frac{250}{5} \rfloor = 50, |A_4| = \lfloor \frac{250}{7} \rfloor = 35$$

$$|A_1 \cap A_2| = \lfloor \frac{250}{2 \times 3} \rfloor = 41, |A_1 \cap A_3| = \lfloor \frac{250}{2 \times 5} \rfloor = 25, |A_1 \cap A_4| = \lfloor \frac{250}{2 \times 7} \rfloor = 17,$$

$$|A_2 \cap A_3| = \lfloor \frac{250}{3 \times 5} \rfloor = 16, |A_1 \cap A_2| = \lfloor \frac{250}{5 \times 7} \rfloor = 7, |A_1 \cap A_2 \cap A_3| = \lfloor \frac{250}{2 \times 3 \times 5} \rfloor = 8,$$

$$|A_1 \cap A_2 \cap A_4| = \lfloor \frac{250}{2 \times 3 \times 7} \rfloor = 5, |A_1 \cap A_3 \cap A_4| = \lfloor \frac{250}{2 \times 3 \times 5} \rfloor = 3$$

$$|A_2 \cap A_3 \cap A_4| = \lfloor \frac{250}{3 \times 5 \times 7} \rfloor = 2, |A_1 \cap A_2 \cap A_3 \cap A_4| = \lfloor \frac{250}{2 \times 3 \times 5 \times 7} \rfloor = 1$$

Hence

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1 = 193$$

Exercise 1.2.1

1. How many natural numbers less than or equal to 1000 or ($n \leq 1000$) are not divisible by any of 2 or 3? (Ans-166)

2. How many natural numbers from 1 through 1000 are multiples of 3 or multiples of 5 ?

Solution 2: Let A – be the set of natural numbers from 1 – 1000 that are multiples of 3
 B – be the set of natural numbers from 1 – 1000 that are multiples of 5.

$A \cup B$ – the set of all natural numbers from 1 to 1000 that are multiples of 3 or multiples of 5 and $A \cap B$ – the set of all natural numbers from 1 to 1000 that are multiples of both 3 and 5 or (multiples of 15).

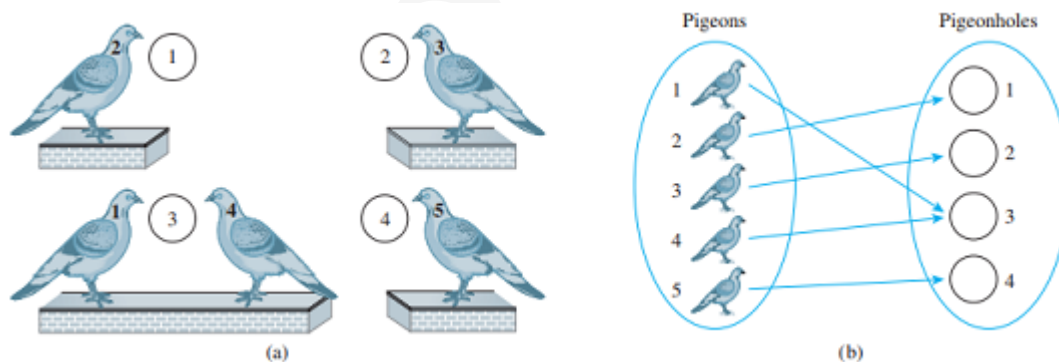
Because every third integer from 3 through 999 is a multiple of 3, each can be represented by $3k$ for some integers from 1 to 333. Hence there are 333 multiple of 3 from 1 through 1000. So $|A| = 333$, Similarly $|B| = 200$ and $|A \cap B| = 66$
 Therefore,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 333 + 200 - 66 = 467 \end{aligned}$$

Hence there 467 natural numbers from 1 to 1000 are multiples of 3 or multiples of 5.

1.3 The Pigeonhole Principle

The pigeonhole principle states that if n pigeons fly into m pigeonholes and $n > m$, then at least one hole must contain two or more pigeons. This principle is illustrated in Figure below, for $n = 5$ and $m = 4$. Illustration (a) shows the pigeons perched next to their holes, and (b) shows the correspondence from pigeons to pigeonholes. The pigeonhole principle is sometimes called the *Dirichlet box principle* because it was first stated formally by J. P. G. L. Dirichlet (1805?1859).



Pigeonhole Principle

A function from one finite set to a smaller finite set cannot be one-to-one: There must be a least two elements in the domain that have the same image in the co-domain.

Theorem 1.3.1 If $n + 1$ or more objects are placed in to n boxes, then there is at least one box containing two or more of the objects.

Corollary 1 A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.

Example 1.3.1 Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Theorem 1.3.2 (Generalized Pigeonhole Principle) If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Example 1.3.2 Among 100 people there are at least $\lceil 100/12 = 9 \rceil$ who were born in the same month.

Exercise 1.3.1

1. How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points? (An: 102)
2. What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F? (An: 26)

1.4 Permutation

Introduction

Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters. For example, in how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Solution: First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5 \times 4 \times 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways to arrange all five students in a line for a picture.

Definition 1.4.1 *Permutation is an ordered arrangement of distinct objects in a row (Any arrangement of a set of n objects in a given order is called a **permutation** of the object (taken all at a time)).*

Example 1.4.1 Given the set of digits $\{1, 2, 3\}$

1. How many two - digit numbers can be formed? (with and without Replacement)
2. How many two - digit numbers can be formed if the number must be even? (with and without Replacement)

3. How many two - digit numbers can be formed if the number must be even and no repetition of digits is allowed?(without Replacement)

Solution

1. Let the two - digit numbers are tens and units (with Replacement)

Tens	Units
3	3

There are 2 places to consider. Each place can be filled by any one of the three digits. Hence there are $3 \times 3 = 9$ different two – digit numbers can be formed.

2. Two digit – number must be even so it must end with 2 (i.e. the last place has one choice) (with Replacement).

Tens	Units
3	1

Therefore, there are $3 \times 1 = 3$ two – digit even numbers.

3. There are two restrictions here, the number must be even and no digit may be repeated.

Tens	Units
2	1

Therefore, there are $2 \times 1 = 2$ two – digit numbers.

Exercise 1.4.1

- How many arrangements of the letters in the word “STUDENT” starts with a vowel? (Ans: 1440)
- Using the digits 2, 4, 6, 8 find how many whole numbers between 3000 and 7000 can be formed in which the digits are all different.(An: 12)
- How many different ways can 12 skiers in the Olympic finals finish the competition? (if there are no ties) And how many different ways can 3 of the skiers finish 1st, 2nd, & 3rd (gold, silver, bronze)?(Ans. 12! and $P(12,3)$ resp.)
- How many different numbers of three digits can be formed from the numbers 1, 2, 3, 4 & 5;
 - If repetitions are allowed?
 - If repetitions are not allowed?
 - How many of these numbers are even in either case?

To find the number of permutations of n objects taken r at a time will be denoted by $p(n, r)$ or nP_r and defined as

$$P(n, r) = \frac{n!}{(n-r)!}, \text{ where } 0 \leq r \leq n$$

a formula which holds also for $r = 0$ and $r = n$ because $P(n, 0) = 1$ and $0! = 1$.

Theorem 1.4.1 The number of permutations of set of size n is given by $P(n, n) = n!$. That is, n elements can be arranged in $n!$ ways.

Example 1.4.2 How many 4 – digit numbers can be formed from the digit 1, 2, 3, ..., 9 if each digit may be used once only?

Solution

Let the four - digit numbers are thousands, hundreds, tens and units

Order of choice	Thousands	Hundreds	Tens	Units
No of possibilities	9	8	7	6

The number of different 4 – digit numbers that can be formed is $9 \times 8 \times 7 \times 6 = 3024$

Using the formula,

$$p(n, r) = \frac{n!}{(n-r)!}$$

$$p(9, 4) = \frac{9!}{(9-4)!} = \frac{9 \times 8 \times 7 \times 6 \times 5!}{5!} = 3024$$

Hence the number of 4 – digit numbers that can be formed is 3024.

Exercise 1.4.2

- Find the value of n if,
 - $p(n, 4) = 42p(n, 2)$
 - $2p(n, 2) + 50 = p(2n, 2)$
- How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?
- How many pairs of dance partners can be selected from a group of 12 women and 20 men?
- A man, a woman, a boy, a girl, a dog, and a cat are walking down a long and winding road one after the other.
 - In how many ways can this happen?
 - In how many ways can this happen if the dog comes first?
 - In how many ways can this happen if the dog immediately follows the boy?
 - In how many ways can this happen if the dog (and only the dog) is between the man and the boy?

1.4.1 Permutation With Repetitions

The number of permutations of a multi set that is, a set of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike is denoted as $p(n; n_1, n_2, \dots, n_r)$ and defined as

$$p(n; n_1, n_2, \dots, n_r) = \frac{n!}{n_1! \times n_2! \times \dots \times n_r!}$$

Example

How many permutations are there of the letters in the word "BENZENE"?

Solution

There are $n = 7$ letters altogether. Of these, 3 are alike (three E's), 2 are alike (two N's), 1 B and 1 Z. Then the permutation of these letters,

$$p(7; 3, 2, 1, 1) = \frac{7!}{3! \times 2! \times 1! \times 1!} = 420$$

1.5 Combinations

Introduction

Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter.

Now, we begin by solving the following question;

How many different committees of three students can be formed from a group of four students?

Solution: To answer this question, we need only find the number of subsets with three elements from the set containing the four students. We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the four students to leave out of the group. This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen does not matter.

Definition 1.5.1 Let S be a set with n elements. A combination of these n elements taken r at a time is any selection of r of the elements where order does not count. Such a selection is called an r -combination; it is simply a subset of S with r elements. The number of such combinations will be denoted by $C(n, r)$, nCr , or $C_{n,r}$ and defined by

$$C(n, r) \text{ or } \binom{n}{r} = \frac{n!}{(n-r)!r!}, \text{ where } 0 \leq r \leq n$$

Example 1.5.1 Find the number of combinations of 4 objects, A, B, C, D, taken 3 at a time. Each combination of three objects determines $3! = 6$ permutations of the objects as follows:

ABC: ABC, ACB, BAC, BCA, CAB, CBA
ABD: ABD, ADB, BAD, BDA, DAB, DBA
ACD: ACD, ADC, CAD, CDA, DAC, DCA
BCD: BDC, BCD, CBD, CDB, DBC, DCB

Thus the number of combinations multiplied by 3! Gives us the number of permutations; that is,

$$C(4, 3) \times 3! = P(4, 3) \text{ or}$$

$$C(4, 3) = \frac{P(4, 3)}{3!}$$

But $P(4, 3) = 4 \times 3 \times 2 = 24$ and $3! = 6$; hence $C(4, 3) = 4$ as noted above.

Therefore, we conclude that:

$$P(n, r) = r! C(n, r) \quad C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}$$

Example 1.5.2 How many ways are there to select 5 players from 10 members to make a trip to a match at another school?

The number of 5 – combinations of a set with 10 elements is

$$C(10, 5) = \frac{10!}{(10-5)!5!} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2} = 252$$

Corollary 2 Let n and r be non-negative integers with $r \leq n$. Then $C(n, r) = C(n, n-r)$ where $0 \leq r \leq n$.

Example 1.5.3 Find the number of groups that can be formed from a group of seven members if each group must contains at least three members.

Solution

Since each group must contain at least three members. It can contain three, four, five, six or seven members. Number of groups containing 3 members $C(7, 3) = 35$, number of groups containing 4 members $C(7, 4) = C(7, 3) = 35$, number of groups containing 5 member $C(7, 5) = 21$, number of groups containing 6 member $C(7, 6) = C(7, 1) = 7$ and number of groups containing 7 member $C(7, 7) = 1$

Hence the total number of groups is $35 + 35 + 21 + 7 + 1 = 99$.

Theorem 1.5.1 $C(n, r) = C(n-1, r-1) + C(n-1, r)$ where $0 < r < n$.

Exercise 1.5.1

1. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is consist of 3 faculty members from the mathematics department and 4 from the computer science department, if there are 9 faculty members of the mathematics department and 11 of the computer Science department?
2. How many committees of five people can be chosen from 20 men and 12 women
 - (a) if exactly three men must be on each committee?
 - (b) if at least four women must be on each committee?

1.5.1 Combinations With Repetition

Before we see formula for combinations with repetition let's Consider the following problem. How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution

To solve this problem we list all the ways possible to select the fruit.

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

Therefore there are 15 ways to select four pieces of fruit from a bowl. The solution is the number of 4-combinations with repetition allowed from a three-element set, {apple, orange, pear}.

Theorem 1.5.2 The number of r – combinations with repetitions from a set of n elements is

$$C(n+r-1, r) \text{ OR } C(n+r-1, n-1)$$

Integer solutions to the linear equation $x_1 + x_2 + \dots + x_n = r$;

- The number of non-negative integer solutions is

$$C(n+r-1, r) \text{ OR } C(n+r-1, n-1)$$

- The number of positive integer solutions is

$$C(r-1, n-1), r \geq n$$

Some combinatorial identities

1. Symmetry Property : $C(n, r) = C(n, n-r)$, where $0 \leq r \leq n$.
2. Newton's Identity : $C(n, r)C(r, k) = C(n, k)C(n-k, r-k)$, $0 \leq k \leq r \leq n$.
3. Pascal's Identity : $C(n, r) = C(n-1, r) + C(n-1, r-1)$, where $0 \leq r \leq n$.
4. $P(n, r) = nP(n-1, r-1)$, $0 \leq r \leq n$.
5. Diagonal Summation

$$C(n, 0) + C(n+1, 1) + C(n+2, 2) + \dots + C(n+r, r) = C(n+r+1, r)$$

6. Row Summation

$$C(n, 0) + C(n, 1) + \dots + C(n, r) + \dots + C(n, n) = 2^n$$

7. Column Summation

$$C(r, r) + C(r+1, r) + \dots + C(n, r) = C(n+1, r+1)$$

$$8. \quad P(n, r) = rP(n-1, r-1) + P(n-1, r), \quad 0 \leq r \leq n.$$

Example 1.5.4 Find the number of non-negative integer solutions of $x + y + z = 18$.

Solution

To find the number of non-negative integer solutions using the above theorem, that is $C(n+r-1, r)$ where $n = 3$, $r = 18$ is $C(n+r-1, r) = C(20, 18) = 190$.

Exercise 1.5.2

1. There are five types of soft drinks at a fast food restaurant: coke classic, diet coke, root beer, Pepsi and Sprite. Find the number of beverage orders 11 guests can make.
2. How many solutions are there of $x + y + z = 17$ in **positive** integers?
3. Evaluate the sum $1 + 2 + 3 + 4 + \dots + n$. (using the combinatorial identities 7)

1.6 Binomial Theorem

As we have seen above, the number of r -combinations from a set with n elements is often denoted by $\binom{n}{r}$. This number is also called a **binomial coefficient** because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a + b)^n$. We will discuss the **binomial theorem**, which gives a power of a binomial expression as a sum of terms involving binomial coefficients

The binomial theorem gives the coefficients of the expansion of powers of binomial expressions. The numbers $\binom{n}{r}$ are called binomial coefficients, since they appear as the coefficients in the expansion of $(x + y)^n$.

Theorem 1.6.1 (Binomial Theorem) Let x and y be variables and n be a positive integer. Then

$$\begin{aligned} (x + y)^n &= \sum_{i=0}^n C(n, i) x^{n-i} y^i \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y^1 + \dots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} y^n \end{aligned}$$

The coefficients of the successive powers of $x + y$ can be arranged in a triangular array of numbers, called **Pascal's triangle**, as pictured in Fig.1. The numbers in Pascal's triangle have the following interesting properties:

1. The first and last number in each row is 1
2. Every other number can be obtained by adding the two numbers appearing above it.

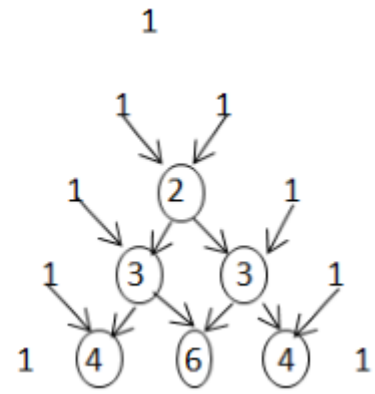
$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$



pascal's triangle

Exercise 1.6.1

1. What is the coefficient of x^2 in the expansion $\left(x^2 + \frac{1}{x}\right)^4$. (Ans. 6)
2. Find the constant term of the expansion $(x + 2/x)^8$. (Ans. 1120)
3. If the sixth term in the expansion of $(\sqrt{x} + 1/x)^n$ is independent of x , then find n . (Ans. 15)

Chapter 2

ELEMENTARY PROBABILITY THEORY

2.1 Introduction

Probability theory is a branch of mathematics that deals with and shows how to measure uncertainty.

Experiment: *is any physically or mentally conceivable (able to be imagined) undertaking that results in a measurable outcome. or An **experiment** is any operation or procedure whose outcomes cannot be predicted with certainty.*

*The set of all possible outcomes for an experiment is called the **sample space** for the experiment.*

*A **fair (unbiased) coin** is a coin that is just as likely to land Heads (H) as it is to land Tails (T).*

Probability: *a numerical value between 0 and 1 measuring the likelihood of occurrence of an event.*

We can do simple experiments to become familiar with the different outcomes that are possible and we can develop and apply some mathematical laws of probability to help us analyze such experiments. We shall often use experiments like tossing coins, rolling dice, drawing cards from a pack and selecting colored balls from a box.

Probability is still used to help people understand games of chance such as black jack carps and lotteries. But there are certainly other uses of probability. Insurance companies are concerned with the probable life expectancy of their policy holders.

Surveyors of public opinion use probability to determine the result of their polls.

Probability is especially important in computer science—In algorithm design and game theory, for example, randomized algorithms and strategies (those that use a random number generator as a key input for decision making) frequently outperform deterministic algorithms and strategies. In information theory and signal processing, an understanding of randomness is critical for filtering out noise and compressing data. In cryptography and digital rights management, probability is crucial for achieving security.

2.2 Sample Space and Events

*When we toss a die, we are performing an experiment to see which set of dots or numbers will turn face up. The number of dots that are on the top surface we toss the die is called the **outcome** of the experiment. Each number has equal chance of occurring so we say that each outcome 1, 2, 3, 4, 5,*

6 is equally likely to occur.

For example, the experiment performing equal size balls in the box differing by color is equally likely an outcome because the balls differ only by color and we can't tell the color of a ball without our eyes.

Definition 2.2.1 The sample space (the possibility set) for an experiment is the set of all possible outcomes of an experiment and denoted by S .

Definition 2.2.2 An Event E is any sub set of the sample space.

Example 2.2.1 Experiment: Toss a coin three times and observe the sequence of heads (H) and tails (T) that appears.

The sample space consists of the following eight elements:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Let A be the event that two or more heads appear consecutively, and B that all the tosses are the same:

$$A = \{HHH, HHT, THH\} \text{ and } B = \{HHH, TTT\}$$

Then $A \cap B = \{HHH\}$ is the elementary event that only heads appear. The event that five heads appears is the empty set $\emptyset$.

2.3 Probability of an Event

Definition 2.3.1 If S is the sample space of an experiment, and if each element of S is equally likely outcomes then the probability of an event E is denoted by $P(E)$ and defined as

$$P(E) = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{|E|}{|S|}$$

For example, the probability of getting two or more heads appear consecutively is

$$P(E) = \frac{|E|}{|S|} = 3/8$$

Properties of Probability

Let P be a probability function on a sample space S , then

1. For every event $E \subseteq S$, $0 \leq P(E) \leq 1$.
2. $P(\emptyset) = 0$, $P(S) = 1$
3. For every event $E \subseteq S$, if $\bar{E}$ is the complement of E (not E) then,

$$P(\bar{E}) = 1 - P(E)$$

4. If $E_1, E_2 \subseteq S$ are two events, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

In particular, if $E_1 \cap E_2 = \emptyset$ (E_1 and E_2 are mutually exclusive) then,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

5. If $A, B \subset S$ and $A \subset B$ then $p(A) \subset p(B)$.

6. For any events A and B , $P(A) = P(A \cap B) + P(A \cap B')$

Example 2.3.1 Find the probability of getting a sum different from 10 or 12 after rolling two dice.

Solution

When we roll two dice, we get the total number of sample space is 36. From the sample space we get 10 in three different ways: 4 + 6, 5 + 5, 6 + 4. Then

$$p(\text{getting a sum 10}) = \frac{\text{number of getting 10}}{\text{sample space}} = 3/36$$

And we can get 12 in one way: 6 + 6;

$$p(\text{getting a sum 12}) = \frac{\text{number of getting 12}}{\text{sample space}} = 1/36$$

Since they are mutually exclusive events, then the probability of getting 10 or 12 is

$$p(\text{getting a sum 10 or 12}) = P(\text{getting a sum 10}) + p(\text{getting a sum 12}) = 3/36 + 1/36 = 1/9.$$

So the probability of not getting 10 or 12 is

$$\begin{aligned} p(\text{not getting a sum 10 or 12}) &= 1 - p(\text{getting a sum 10 or 12}) \\ &= 1 - 1/9 = 8/9. \end{aligned}$$

Exercise 2.3.1

1. One integer is chosen at random from the set of integers from 1 to 50 inclusive. What is the probability that it is
 - (a) Not prime. (Ans. 7/10)
 - (b) at least 4. (Ans. 47/50)
2. Five marbles are drawn at random from a bag of seven green marbles and four red marbles. Find the probability that three are green and two are red. (Ans. 5/11)

2.4 Conditional Probability

Suppose that we pick a random person in the world. Everyone has an equal chance of being selected. Let A be the event that the person is an MIT student, and let B be the event that the person lives in Cambridge. What are the probabilities of these events? Intuitively, we're picking a random point in the big ellipse shown in Fig.2 and asking how likely that point is to fall into region A or B .

The vast majority of people in the world neither live in Cambridge nor are MIT students, so events A and B both have low probability. But what about the probability that a person is an MIT student given that the person lives in Cambridge?

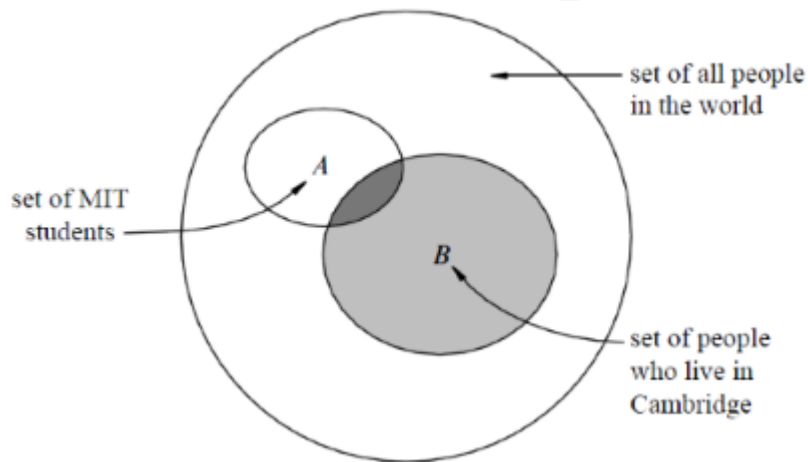


Fig.2. Selecting a random person. A is the event that the person is an MIT student. B is the event that the person lives in Cambridge.

What we're asking for is called a conditional probability; that is, the probability that one event happens, given that some other event definitely happens. Questions about conditional probabilities come up all the time:

- What is the probability that it will rain this afternoon, given that it is cloudy this morning?
- What is the probability that two rolled dice sum to 10, given that both are odd?

There is a special notation for conditional probabilities. In general, $P(A/B)$ denotes the probability of event A , given that event B happens. So, in our example, $P(A/B)$ is the probability that a random person is an MIT student, given that he or she is a Cambridge resident.

How do we compute $P(A/B)$? Since we are given that the person lives in Cambridge, we can forget about everyone in the world who does not. Thus, all outcomes outside event B are irrelevant. So, intuitively, $P(A/B)$ should be the fraction of Cambridge residents that are also MIT students; that is, the answer should be the probability that the person is in set $A \cap B$ (the darkly shaded region in Fig.2) divided by the probability that the person is in set B (the lightly shaded region). This motivates the definition of conditional probability:

Definition 2.4.1 The conditional probability of an event E given that $F \neq \emptyset$ denoted by $P(E/F)$ is the probability of E assuming that F has occurred (it becomes the new sample space replacing the original S) and defined as

$$P(E/F) = \frac{P(E \cap F)}{P(F)}$$

Example 2.4.1 Find the probability of obtaining a sum of 10 after rolling two dice. Find the probability of that event if we know that at least one of the dice shows 5 points.

Solution

Let E be an event obtaining sum 10 and F be an event at least one of the dice shows 5 points. The total number of possible outcomes is 36.

$$E = \{(4, 6), (5, 5), (6, 4)\} \text{ and } E \cap F = \{(5, 5)\}$$

$$F = \{(1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 6)\}$$

$$\text{Then probability of } E \text{ i.e. } P(E) = \frac{|E|}{|S|} = 3/36$$

$$\text{And } |F| = 11, |E| = 3 \text{ and } |E \cap F| = 1 \text{ then } P(E/F) = \frac{P(E \cap F)}{P(F)} = 1/11$$

Example 2.4.2 What probabilities should we assign to the outcomes H (heads) and T (tails) when a fair coin is flipped? What probabilities should be assigned to these outcomes when the coin is biased so that heads comes up twice as often as tails?

Solution

For a fair coin, the probability that heads comes up when the coin is flipped equals the probability that tails comes up, so the outcomes are equally likely. Consequently, we assign the probability $1/2$ to each of the two possible outcomes, that is, $p(H) = p(T) = 1/2$. For the biased coin we have

$$P(H) = 2P(T)$$

Because

$$P(H) + P(T) = 1$$

it follows that

$$2P(T) + P(T) = 3P(T) = 1$$

We conclude that $p(T) = 1/3$ and $p(H) = 2/3$.

Exercise 2.4.1

1. Pair of fair dice is tossed. The sample space S consists of the 36 ordered pairs (a, b) , where a and b can be any of the integers from 1 to 6. Thus the probability of any point is $1/36$. Find the probability that one of the dice is 2 if the sum is 6. (Ans. $2/5$)
2. A couple has two children; the sample space is $S = \{bb, bg, gb, gg\}$ with probability $1/4$ for each point. Find the probability p that both children are boys if it is known that:
 - (a) at least one of the children is a boy; (Ans. $1/3$)
 - (b) the older child is a boy. (Ans. $1/2$)

2.5 Independent Events

Events A and B in a probability space S are said to be independent events if the occurrence of one of them does not influence the occurrence of the other. More specifically, the event B is independent of event A if $P(B)$ is the same as $P(B|A)$.

Now substituting $P(B)$ for $P(B|A)$ in $P(A \cap B) = P(A)P(B|A)$ yields

$$P(A \cap B) = P(A)P(B).$$

Definition 2.5.1 If A and B are events in a sample space S , then A and B are independent events if, and only if

$$P(A \cap B) = P(A)P(B)$$

For instance, suppose that we flip two fair coins simultaneously on opposite sides of a room. Intuitively, the way one coin lands does not affect the way the other coin lands. The mathematical concept that captures this intuition is called **independence**.

- We say that three events A_1, A_2, A_3 are independent if they are **pairwise independent**. that is,

$$P(A_j \cap A_k) = P(A_j)P(A_k), j \neq k \text{ where } j, k = 1, 2, 3$$

and

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

Example 2.5.1 A fair coin is tossed three times yielding the equiprobable space

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Consider the events:

$$A = \{\text{first toss is heads}\} = \{HHH, HHT, HTH, HTT\}$$

$$B = \{\text{second toss is heads}\} = \{HHH, HHT, THH, THT\}$$

$$C = \{\text{exactly two heads in a row}\} = \{HHT, THH\}$$

Clearly A and B are independent events; this fact is verified below. On the other hand, the relationship between A and C and between B and C is not obvious. We claim that A and C are independent, but that B and C are dependent. We have:

$$P(A) = 1/2, \quad P(B) = 1/2, \quad P(C) = 1/4$$

Also

$$P(A \cap B) = 1/4, \quad P(A \cap C) = 1/8, \quad P(B \cap C) = 1/4$$

Then, $P(A)P(B) = 1/2 \times 1/2 = 1/4 = P(A \cap B)$ So, A and B are independent

$$P(A)P(C) = 1/2 \times 1/4 = 1/8 = P(A \cap C) \text{ So, } A \text{ and } C \text{ are independent}$$

$$P(B)P(C) = 1/2 \times 1/4 = 1/8 \neq P(B \cap C) \text{ So, } B \text{ and } C \text{ are dependent.}$$

2.6 Random Variable and Expectations

Let S be a sample of an experiment. The outcome of the experiment, or the points in S , need not be numbers. For example, in tossing a coin the outcomes are H (heads) or T (tails), and in tossing a pair of dice the outcomes are pairs of integers. However, we frequently wish to assign a specific number to each outcome of the experiment. For example, in coin tossing, it may be convenient to assign 1 to H and 0 to T; or, in the tossing of a pair of dice, we may want to assign the sum of the two integers to the outcome. Such an assignment of numerical values is called a random variable.

Definition 2.6.1 A random variable is a function X from the sample space S of an experiment to the set of real numbers. (That is a random variable assigns a real number to each possible outcomes.)

Example 2.6.1 Suppose that a coin is flipped three times. Let $X(t)$ be the number of heads that appear when t is the outcome. Then the random variable $X(t)$ takes the following values:

$$X(HHH) = 3$$

$$X(HHT) = X(HTH) = X(THH) = 2$$

$$X(HTT) = X(THT) = X(TTH) = 1$$

$$X(TTT) = 0$$

Exercise 2.6.1

1. Let X be the sum of the numbers that appear when two dice are rolled. What are the values of the random variable for the 36 possible outcomes (i, j) where i and j are the numbers that appear on the first die and the second die respectively.

Probability Distribution of a Random Variable

Let X be a random variable on a finite sample space S with range space $R_x = \{x_1, x_2, \dots, x_t\}$. Then X induces a function f which assigns probabilities P_k to the points x_k in R_x as follows:

$f(x_k) = P_k = P(X = x_k) = \text{Sum of probabilities of points in } S \text{ whose image is } x_k.$

The set of ordered pairs $(x_1, f(x_1)), (x_2, f(x_2)), \dots, (x_t, f(x_t))$, is called the distribution of the random variable X . This function f has the following two properties:

$$\begin{aligned} & \bullet f(x_k) \geq 0 \\ & \bullet \sum_k f(x_k) = 1 \end{aligned}$$

Theorem 2.6.1 Let S be an equiprobable space, and let f be the distribution of a random variable X on S with the range space $R_x = \{x_1, x_2, \dots, x_t\}$. Then

$$P_i = f(x_i) = \frac{\text{number of points in } S \text{ whose image is } x_i}{\text{number of points in } S}$$

Definition 2.6.2 The expected value (expectation) denoted by E of the random variable $X(s)$ on the sample space S is defined as

$$E(X) = \sum_{s \in S} P(s)X(s)$$

Note that when the sample space S has n elements $S = \{x_1, x_2, \dots, x_n\}$ then

$$E(X) = \sum_{i=1}^n P(x_i)X(x_i)$$

Example 2.6.2

1. A fair coin is flipped three times. Let S be the sample space of the eight possible outcomes, and let X be the random variable that assigns to the number of heads in the outcome. What is the expected value of X ?

Solution

The values of random variable X for each the eight possible outcomes are 3, 2, 2, 2, 1, 1, 1, 0 and the probability of each outcome is $1/8$. Then

$$E(X) = \sum_{i=1}^n P(x_i)X(x_i)$$

$$E(X) = 1/8(3 + 2 + 2 + 2 + 1 + 1 + 1 + 0) = 3/2$$

2. Let X be the number that comes up when a fair die is rolled. What is the expected value of X ?

Solution

Let X takes the values 1, 2, 3, 4, 5, 6 and the probability of each outcome is $1/6$.

$$\begin{aligned} E(X) &= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) \\ &= \frac{7}{2} \end{aligned}$$

Exercise 2.6.2

1. What is the expected value of the sum of numbers that appear when a pair of fair dice is rolled? (Ans. 7)

Theorem 2.6.2 If X and Y are random variables on a space S , then $E(X + Y) = E(X) + E(Y)$. Furthermore, if a and b are real numbers then $E(aX + b) = aE(X) + b$.

Proof

From the definition of expected value

$$\begin{aligned} E(X + Y) &= \sum_{s \in S} P(s)(X(s) + Y(s)) \\ &= \underbrace{\sum_{s \in S} P(s)X(s)}_{E(X)} + \underbrace{\sum_{s \in S} P(s)Y(s)}_{E(Y)} \\ &= E(X) + E(Y) \end{aligned}$$

And

$$\begin{aligned} E(aX + b) &= \sum_{s \in S} P(s)(aX(s) + b) \\ &= \sum_{s \in S} P(s)aX(s) + \sum_{s \in S} P(s)b \\ &= a \underbrace{\sum_{s \in S} P(s)X(s)}_{=E(X)} + b \underbrace{\sum_{s \in S} P(s)}_{=1} \\ &= aE(X) + b \blacksquare \end{aligned}$$

2.6.1 Independent random variables

Definition 2.6.3 The random variables X and Y on the sample space S are independent if

$$P(X(s) = r_1 \text{ and } Y(s) = r_2) = P(X(s) = r_1) \times P(Y(s) = r_2)$$

Theorem 2.6.3 If X and Y are two independent random variables on the space S , then

$$E(XY) = E(X)E(Y)$$

Proof

From definition of expected value and since X and Y are independent random variables;

$$\begin{aligned} E(XY) &= \sum_{s \in S} X(s)Y(s)P(s) \\ &= \sum_{r_1 \in X(s), r_2 \in Y(s)} r_1 r_2 P(X(s) = r_1 \text{ and } Y(s) = r_2) \\ &= \underbrace{\left(\sum_{r_1 \in X(s)} r_1 P(X(s) = r_1) \right)}_{=E(X)} \underbrace{\left(\sum_{r_2 \in Y(s)} r_2 P(Y(s) = r_2) \right)}_{=E(Y)} \\ &= E(X)E(Y) \blacksquare \end{aligned}$$

Chapter 3

RECURRENCE RELATIONS

3.1 Introduction

Many counting problems cannot be solved easily using the techniques discussed in previous Chapters . One such problem is: For example, suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in n hours? Before we are going to solve this problem; first let's see some terminologies:

Definition 3.1.1 A sequence is a function whose domain is natural number.(i.e A sequence is a function $f : N \longrightarrow \mathbb{R}$, whose domain is natural number.)

Example 3.1.1

Let's consider $a_n = 2n - 1$, $n \in N$ and $a_n = 1/n$, $n \in N$ are examples of sequences. The terms of the sequence $a_n = 2n - 1$, $n \in N$ are

$$\{a_n\} = \{2n - 1\}_{n=1}^{\infty} = \{1, 3, 5, \dots\}$$

Definition 3.1.2 A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, \dots, a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a non-negative integer.

A sequence is called a **solution** of a recurrence relation if it satisfies the recurrence relation.

Example 3.1.2 The following are an examples of recurrence relations:

1. $a_n = a_{n-1} - a_{n-2}$
2. $a_n - 2a_{n-1} - a_{n-2} = n^2 + 6$
3. $a_n + (a_{n-1})^n + a_{n-2} = 6$
4. $a_n + n^3 + 2 = 0$
5. $a_{n+3} + 5a_{n+2} + 4a_{n+1} + a_n = \cos n$
6. $a_{n+1}^4 + a_n^5 = n$

Example 3.1.3 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$, $n \geq 2$ With $a_0 = 3$, $a_1 = 5$ then find a_2, a_3 .

Solution: Since the recurrence relation given by

$$a_n = a_{n-1} - a_{n-2}$$

then the second term a_2 when $n = 2$ is obtained as

$$\begin{aligned} a_2 &= a_1 - a_0 \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

And

$$\begin{aligned} a_3 &= a_2 - a_1 \\ &= 2 - 5 \\ &= -3 \end{aligned}$$

Hence $a_2 = 2$ and $a_3 = -3$.

Example 3.1.4

1. Show that $a_n = 5^n$, $n \geq 0$ is a solution of the recurrence relation $a_n = 5a_{n-1}$ for $n \geq 1$.

Solution: We have to check that the sequence $a_n = 5^n$ satisfies the given relation:

$$5a_{n-1} = 5 \times 5^{n-1} = 5 \times 5^n \times 1/5 = 5^n = a_n$$

Hence $a_n = 5^n$ is the solution of the recurrence relation.

2. If c_1 and c_2 are arbitrary constants, then show that $a_n = c_1 2^n + c_2 5^n$ is a solution of the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $n \geq 2$

Solution: Let's check the sequence $a_n = c_1 2^n + c_2 5^n$ satisfy the recurrence relation

$$\begin{aligned} a_n - 7a_{n-1} + 10a_{n-2} &= (c_1 2^n + c_2 5^n) - 7(c_1 2^{n-1} + c_2 5^{n-1}) + 10(c_1 2^{n-2} + c_2 5^{n-2}) \\ &= c_1 2^n - 7c_1 2^{n-1} + 10c_1 2^{n-2} + c_2 5^n - 7c_2 5^{n-1} + 10c_2 5^{n-2} \\ &= 2^{n-2} c_1 (2^2 - 7(2) + 10) + c_2 5^{n-2} (5^2 - 7(5) + 10) \\ &= 2^{n-2} c_1 (0) + c_2 5^{n-2} (0) = 0 \end{aligned}$$

Hence $a_n = c_1 2^n + c_2 5^n$ is the solution of the recurrence relation

$$a_n - 7a_{n-1} + 10a_{n-2} = 0.$$

Exercise 3.1.1

1. For arbitrary constants c_1 and c_2 , check that $a_n = c_1 5^n + c_2 n 5^n$ satisfies the recurrence relation $a_n - 10a_{n-1} + 25a_{n-2} = 0$.
2. Show that the sequence $a_n = 2n + c$ is the solution of the recurrence relation $a_n - a_{n-1} = 2$.

Modeling With Recurrence Relations

We can use recurrence relations to model a wide variety of problems, such as number of bacteria in a colony, counting rabbits on an island, determining the number of moves in the Tower of Hanoi puzzle, and counting bit strings with certain properties.

Example 3.1.5 Suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in n hours?

To solve this problem, let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer. This recurrence relation, together with the initial condition $a_0 = 5$, uniquely determines a_n for all non-negative integers n . We get a solution of the RR with $a_n = 5 \cdot 2^n$ for all non-negative integers n .

Example 3.1.6 A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month, as shown in Figure 3. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Rabbits on an Island

Solution: let f_n be the number of pairs of rabbits after n months. We will show that $f_n, n = 1, 2, 3, \dots$, are the terms of the sequence. The rabbit population can be modeled using a recurrence relation. At the end of the first month, the number of pairs of rabbits on the island is $f_1 = 1$. Because this pair does not breed during the second month, $f_2 = 1$. To find the number of pairs after n months, add the number on the island the previous month, f_{n-1} , and the number of newborn pairs, which equals f_{n-2} , because each newborn pair comes from a pair at least 2 months old. Consequently, the sequence $\{f_n\}$ satisfies the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3$$

Together with the initial conditions $f_1 = 1$ and $f_2 = 1$.

Exercise 3.1.2

- Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires 1 microsecond, and the transmittal of the other signal requires 2 microseconds.
 - Find a recurrence relation for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, which can be sent in n microseconds.
 - What are the initial conditions?
 - How many different messages can be sent in 10 microseconds using these two signals?
- Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length five?

Definition 3.1.3 Suppose that n and k are non-negative integers. A linear recurrence relation with constant coefficient of degree k is a recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n), \text{ for } n \geq k$$

Where $c_0, c_1, \dots, c_k$ are constants, c_0 and c_k are not identically zero and $f(n)$ are functions of n .

Example 3.1.7

1. The recurrence relation $a_n = 2a_{n-1}$ is a linear recurrence relation with constant coefficient of degree 1.
2. The recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-6} + n + 1$ is linear recurrence relation with constant coefficient of degree 6.
3. The recurrence relation $a_n - 3(a_{n-1})^2 + 2a_{n-2} = n$ is not linear since the power of a_{n-1} is 2.
4. $a_n = na_{n-1}$ is linear recurrence relation but does not have constant coefficients.

Definition 3.1.4 A homogeneous linear recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = 0, \quad n \geq k$$

where $c_1, c_1, \dots, c_k$ are arbitrary constants $c_k \neq 0$.

Definition 3.1.5 A non-homogeneous linear recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n), \quad n \geq k$$

where $c_1, c_1, \dots, c_k$ are arbitrary constants $c_k \neq 0$ and $f(n) \neq 0$.

Example 3.1.8

1. The recurrence relation $a_n = 2a_{n-1}$ is a homogeneous linear recurrence relation of degree 1.
2. In the above example, a & d are homogeneous, and b & c are non-homogeneous recurrence relation.

3.2 Solving Recurrence Relations; The Characteristic Polynomial

3.2.1 Solving Homogeneous Linear RR with Constant Coefficients

For solving linear homogeneous recurrence relation with constant coefficients is to look for solutions of the form $a_n = r^n$ where r is constant. Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

When both sides divided by r^{n-k} , we obtain the equivalent equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

This equation is called **characteristic equation (polynomial)** of the recurrence relation, and then the sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution of the characteristic equation. The solutions of the characteristic equation are called the characteristic roots of the recurrence relation, and then the characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

We now turn our attention to linear homogeneous recurrence relations of degree two. First, consider the case when there are two distinct characteristic roots.

Theorem 3.2.1 Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2 . Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \dots$ where α_1 and α_2 are constants.

Proof:

The proof of this theorem consists of two parts:

- First, we will show that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation for any constants α_1 and α_2 .
- Second, it must be shown that if the sequence $\{a_n\}$ is a solution, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants α_1 and α_2 .

Now we will show that if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, then the sequence $\{a_n\}$ is a solution of the recurrence relation. Because r_1 and r_2 are roots of $r^2 - c_1r - c_2 = 0$, it follows that $r_1^2 = c_1 r_1 + c_2$, $r_2^2 = c_1 r_2 + c_2$.

From these equations, we see that

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1 (\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2 (\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2} (c_1 r_1 + c_2) + \alpha_2 r_2^{n-2} (c_1 r_2 + c_2) \\ &= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\ &= \alpha_1 r_1^n + \alpha_2 r_2^n \\ &= a_n \end{aligned}$$

This shows that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation.

Secondly, To show that every solution $\{a_n\}$ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \dots$, for some constants α_1 and α_2 , suppose that $\{a_n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold. It will be shown that there are constants α_1 and α_2 such that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies these same initial conditions. This requires that

$$\begin{aligned} a_0 &= C_0 = \alpha_1 + \alpha_2 \\ a_1 &= C_1 = \alpha_1 r_1 + \alpha_2 r_2 \end{aligned}$$

We can solve these two equations for α_1 and α_2 . From the first equation it follows that $\alpha_2 = C_0 - \alpha_1$. Inserting this expression into the second equation gives

$$\begin{aligned} C_1 &= \alpha_1 r_1 + (C_0 - \alpha_1) r_2 \\ &= \alpha_1 (r_1 - r_2) + C_0 r_2 \end{aligned}$$

This shows that

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2} \text{ and } \alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2} \text{ provided that } r_1 \neq r_2$$

With these value for α_1 and α_2 , a_n satisfies the initial conditions and the recurrence relation. Since the recurrence relation and initial conditions determine a unique sequence, a_n , $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is indeed the unique solution of the recurrence relation.

The solutions r_1^n and r_2^n are the **basic solutions** of the recurrence relation. In general, the number of basic solutions equals the order of the recurrence relation. The **general solution** $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a **linear combination** of the basic solutions.

Example 3.2.1 What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

Solution: In order to find the characteristic equation of the recurrence relation, take $a_n = r^n$ to the recurrence relation:

$$a_n = a_{n-1} + 2a_{n-2}$$

$$a_n - a_{n-1} - 2a_{n-2} = 0$$

$$r^n - r^{n-1} - 2r^{n-2} = 0$$

$r^2 - r - 2 = 0$ is the characteristic equation of the recurrence relation.

$$(r - 2)(r + 1) = 0$$

$r_1 = 2$ and $r_2 = -1$ are the roots of the characteristic equation. Hence the sequence $\{a_n\}$ is a general solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n \text{ for some constants } \alpha_1, \alpha_2$$

From the initial conditions we have to find the values of α_1 and α_2 ;

$$a_0 = \alpha_1 2^0 + \alpha_2 (-1)^0$$

implies

$$2\alpha_1 + \alpha_2 = 2 \quad (a)$$

And

$$a_1 = \alpha_1 2^1 + \alpha_2 (-1)^1 = 7$$

$$2\alpha_1 - \alpha_2 = 7 \quad (b)$$

From (a) and (b), we get

$$\alpha_1 = 3 \text{ and } \alpha_2 = -1.$$

Hence, the solution to the recurrence relation with the initial conditions is the sequence $\{a_n\}$ is $a_n = 3 \times 2^n - 1(-1)^n$.

Exercise 3.2.1

1. Solve the following recurrence relation together with the initial conditions given

(a) $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 0, a_1 = 1$.

(b) $a_n = 5a_{n-1} - 6a_{n-2}, n > 2$, given $a_0 = 1, a_1 = 4$.

(c) $a_n = 2a_{n-1} + 3a_{n-2}, n > 2$, given $a_0 = -2, a_1 = 1$.

(d) $a_n = a_{n-1} + 6a_{n-2}, n > 2$, given $a_0 = 1, a_1 = 3$.

(e) $a_n = 4a_{n-2}$ for $n \geq 2, a_0 = 0, a_1 = 4$.

Theorem 3.2.2 Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that the characteristic polynomial $r^2 - c_1 r - c_2 = 0$ has only one root r . Then the sequence $\{a_n\}$ is a general solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r^n + \alpha_2 n r^n$ for $n = 0, 1, 2, \dots$ where α_1 and α_2 are constants.

Example 3.2.2 Solve $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1, a_1 = 6$.

Solution

The characteristic polynomial of the recurrence relation is

$$r^2 - 6r + 9 = 0$$

Then $r = 3$ is the only root of the recurrence relation. Therefore the solution to the recurrence relation is $a_n = \alpha_1 3^n + \alpha_2 n 3^n$ for some constants α_1 and α_2 . And again using the initial conditions find the values of the arbitrary constants;

$$a_0 = 1 \implies a_0 = \alpha_1 \times 3^0 + \alpha_2 \times 0 \times 3^0 = 1$$

$$\alpha_1 = 1$$

$$a_1 = 6 \implies \alpha_1 \times 3^1 + \alpha_2 \times 1 \times 3^1 = 6$$

$$\implies 3\alpha_1 + 3\alpha_2 = 6$$

$$\implies 3 + 3\alpha_2 = 6, \text{ then } \alpha_2 = 1$$

Hence the solution to the recurrence relation with the initial conditions is $a_n = 3^n + n3^n$.

Exercise 3.2.2

1. Let $a, b \in \mathbb{R}$ and $b \neq 0$. Let α be a real or complex solution of the equation $r^2 - ar - b = 0$ with degree of multiplicity two. Then find the general solution of the recurrence relation $a_n = aa_{n-1} + ba_{n-2}$.
2. Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$, $n > 2$, with initial conditions $a_0 = 1, a_1 = 4$.
3. Solve the recurrence relation $a_{n+1} = -8a_n - 16a_{n-1}$, $n > 1$, given $a_0 = 5, a_1 = 17$.

Theorem 3.2.3 Let $c_1, c_2, \dots, c_k$ and $c_k \neq 0$ be real numbers. Suppose that the characteristic polynomial $r^k - c_1 r^{k-1} - \dots - c_k = 0$ has k distinct roots $r_1, r_2, \dots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$ for $n = 0, 1, 2, \dots$ where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

Example 3.2.3 Find the solution of the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial conditions $a_0 = 2, a_1 = 5$ and $a_2 = 15$.

Solution

The characteristic equation of this recurrence relation is $r^3 - 6r^2 + 11r - 6 = 0$, then factorize this polynomial equation of degree 3 as ; $(r-1)(r-2)(r-3) = 0 \implies r_1 = 1, r_2 = 2$ and $r_3 = 3$ are the roots. Hence the solution to this recurrence relation is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$.

To find the constants α_1, α_2 and α_3 use the initial conditions;

$$a_0 = 2 \implies \alpha_1 + \alpha_2 + \alpha_3 = 2$$

$$a_1 = 5 \implies \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$$

$$a_2 = 15 \implies \alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$$

When those three simultaneous equations are solved for α_1, α_2 and α_3 we obtain, $\alpha_1 = 1, \alpha_2 = -1$ and $\alpha_3 = 2$. Hence the solution to this recurrence relation with the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1^n - 2^n + 2 \cdot 3^n$$

Exercise 3.2.3 Find the general solution of the recurrence relation $a_n - 5a_{n-1} + 2a_{n-2} + 8a_{n-3} = 0$.

Theorem 3.2.4 Let $c_1, c_2, \dots, c_k$ and $c_k \neq 0$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \dots - c_k = 0$ has only one root r . Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ if and only if $a_n = \alpha_1 r^n + \alpha_2 n r^n + \dots + \alpha_k n^{k-1} r^n$ for $n = 0, 1, 2, \dots$ where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

Theorem 3.2.5 Let $c_1, c_2, \dots, c_k$ and $c_k \neq 0$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \dots - c_k = 0$ has t distinct roots $r_1, r_2, \dots, r_t$ with multiplicities $m_1, m_2, \dots, m_t$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \dots, t$ and $m_1 + m_2 + \dots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n \\ + (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n \\ + \dots + (\alpha_{t,0} + \alpha_{t,1}n + \dots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n$$

For $n = 0, 1, 2, \dots$ where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Example 3.2.4 Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1, a_1 = -2$ and $a_2 = -1$.

Solution

The characteristic equation is

$$r^3 + 3r^2 + 3r + 1 = 0 \\ (r + 1)^3 = 0$$

Since the characteristic equation has one root $r = -1$ with multiplicity 3 to find its solution we can use theorem 3.2.4 or theorem 3.2.5. Let's take the last theorem; the solution to this recurrence relation is of the form

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)r^n$$

To find the constants $\alpha_{1,0}, \alpha_{1,1}$ and $\alpha_{1,2}$ using the initial conditions;

$$a_0 = 1 \implies (\alpha_{1,0} + \alpha_{1,1} \times 0 + \alpha_{1,2} \times 0^2)(-1)^0 = 1$$

$$\alpha_{1,0} = 1$$

$$a_1 = -2 \implies (\alpha_{1,0} + \alpha_{1,1} \times 1 + \alpha_{1,2} \times 1^2)(-1)^1 = -2$$

$$-\alpha_{1,1} - \alpha_{1,2} = -1$$

$$a_2 = -1 \implies (\alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2})(-1)^2 = -1$$

$$2\alpha_{1,1} + 4\alpha_{1,2} = -2$$

From these equations $\alpha_{1,0} = 1, \alpha_{1,1} = 3$ and $\alpha_{1,2} = -2$. hence a solution to this recurrence relation with the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = (1 + 3n - 2n^2)(-1)^n$$

Exercise 3.2.4

1. Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5, a_1 = 4$, and $a_2 = 88$.
2. Find the general solution of the recurrence relation $a_n + 2a_{n-1} - 3a_{n-2} - 4a_{n-3} + 4a_{n-4} = 0$.
3. Solve the recurrence relation $a_n = 7a_{n-1} - 13a_{n-2} - 3a_{n-3} + 18a_{n-4}$, where $a_0 = 5, a_1 = 3, a_2 = 6$ and $a_3 = -21$.

3.2.2 Non-homogeneous linear recurrence relation with constant coefficients

Non-homogeneous linear recurrence relation with constant coefficients is a recurrence relation of the form

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n), \quad n \geq k \quad (*)$$

where $c_1, c_1, \dots, c_k, c_k \neq 0$ are arbitrary constants and $f(n) \neq 0$. The recurrence relation

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$$

is called the associated homogeneous recurrence relation of (*).

To solve non-homogeneous linear recurrence relation with constant coefficients we have the following steps:

1. Find solution of the associated homogeneous recurrence relation, but don't yet use the initial conditions to determine coefficients.
2. Find the particular solution to the non-homogeneous linear recurrence, ignoring initial conditions. We'll explain how to find a particular solution next.
3. Add the solution to the associated homogeneous $a_n^{(h)}$ and particular solutions $a_n^{(p)}$ together to obtain the general solution.

$$a_n = a_n^{(h)} + a_n^{(p)}$$

4. Now use the initial conditions to determine the constants by the usual method of generating and solving a system of linear equations.

To find a particular solution of the recurrence relation by guessing as in the form of:

$f(n)$	Form of particular solution
a constant c	a constant d
a linear function $c_0 + c_1 n$	a linear function $d_0 + d_1 n$
an m^{th} degree polynomial $c_0 + c_1 n + \cdots + c_m n^m$	an m^{th} degree polynomial $c_0 + c_1 n + \cdots + c_m n^m$
an exponential form ca^n , a is constant	an exponential form da^n a is constant
$c \sin \alpha n$	$d_1 \sin \alpha n + d_2 \cos \alpha n$
$c \cos \alpha n$	$d_1 \sin \alpha n + d_2 \cos \alpha n$
$ca^n \cos \alpha n$	$d_1 a^n \cos \alpha n + d_2 a^n \sin \alpha n$

Note

There is no general procedure for determining the particular solution of a recurrence relation. So the solution can be obtained only by the method of inspection.

Example 3.2.5 Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n$ where $a_0 = 1$ and $a_1 = 2$.

Solution

The associated homogeneous equation is

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

The characteristic polynomial to the associated homogeneous recurrence is

$$r^2 - 7r + 10 = 0$$

$$(r - 2)(r - 5) = 0 \implies r_1 = 2, \quad r_2 = 5$$

The homogeneous solution is $a_n^h = \alpha_1 2^n + \alpha_2 5^n$. And since $f(n)$ is a linear polynomial so, the particular solution is of the form $a_n^p = d_0 + d_1 n$ substituting a_n^p in the recurrence relation we get;

$$(d_0 + d_1 n) - 7(d_0 + d_1(n-1)) + 10(d_0 + d_1(n-2)) = 8n + 6$$

$$4d_0 - 13d_1 + 4nd_1 = 8n + 6$$

$$4d_0 - 13d_1 = 6 \text{ and } 4nd_1 = 8n$$

From this we get, $d_1 = 2$ and $d_0 = 8$ therefore, $a_n^p = d_0 + d_1 n = 8 + 2n$

The general solution is $a_n = a_n^h + a_n^p = \alpha_1 2^n + \alpha_2 5^n + 2n + 8$, to find α_1, α_2 use the initial conditions $a_0 = 1$ and $a_1 = 2$ we get;

$$\begin{cases} \alpha_1 + \alpha_2 = -7 \\ 2\alpha_1 + 5\alpha_2 = -8 \end{cases}$$

After we solve this simultaneous equation we get $\alpha_1 = -9$ and $\alpha_2 = 2$. Hence $a_n = -9 \times 2^n + 2 \times 5^n + 2n + 8$.

Exercise 3.2.5

1. Solve $a_n = 3a_{n-1} + 2n$ with $a_0 = 3$.
2. Solve $a_n = 5a_{n-1} - 6a_{n-2} + 8n^2$, where $a_0 = 4$ and $a_1 = 7$.

Remark 2 Base On Right Hand Side Equal To Characteristic Root

If the Right hand side of

$$a_n + c_1 a_{n-1} + \dots + c_k a_{n-k} = f(n)$$

involves an exponential with base r and " r " is the characteristic root with multiplicity m , then multiply the guessed form of particular solution by n^m where n is the index of the sequence.

Example 3.2.6 Solve the recurrence $a_n - 6a_{n-1} + 9a_{n-2} = 3^{n+1}$.

Solution

The characteristic equation of the associated linear homogeneous recurrence relation is

$$r^2 - 6r + 9 = 0$$

$$(r - 3)^2 = 0$$

$r = 3$ is the only characteristic root with multiplicity 2. then the particular solution has the form

$$a_n^p = dr^n n^2 = dn^2 3^n$$

Example 3.2.7 Solve the recurrence $a_n - a_{n-1} - 12a_{n-2} = (-3)^n + 4^n$.

Solution

The particular solution has the form

$$a_n^p = d_1 r_1^n n + d_2 r_2^n n = d_1 (-3)^n n + d_2 4^n n$$

3.3 Solving Recurrence Relations; Generating Functions

Introduction

Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series. Generating functions can be used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints, and the number of ways to make change for a dollar using coins of different denominations. Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function. This equation can then be solved to find a closed form for the generating function. From this closed form, the coefficients of the power series for the generating function can be found, solving the original recurrence relation.

Definition 3.3.1 Let $a_0, a_1, a_2, \dots$ be a sequence of real numbers. Then the function

$$g(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \quad (3.1)$$

is the **generating function** for the sequence a_n . Generating functions for the finite sequence $a_0, a_1, \dots, a_n$ can also be defined by letting $a_i = 0$ for $i > n$; thus $g(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is the generating function for the finite sequence $a_0, a_1, \dots, a_n$.

Example 3.3.1 The generating function of the sequence $1, 2, 3, \dots$ of natural numbers is $f(x) = 1 + 2x + 3x^2 + \dots$, while the generating function of the arithmetic sequence $1, 4, 7, 10, \dots$ is $f(x) = 1 + 4x + 7x^2 + 10x^3 + \dots$.

Since

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \dots + x^{n-1}$$

Then $g(x) = \frac{x^n - 1}{x - 1}$ is the generating function for the sequence of n ones.

Addition and Multiplication of Generating Functions

Let $f(x) = \sum_{n=0}^{\infty} a_nx^n$ and $g(x) = \sum_{n=0}^{\infty} b_nx^n$ be two generating functions. Then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)x^n \quad \text{and} \quad f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^n a_i b_{n-i} \right) x^n$$

For example,

$$\begin{aligned} \frac{1}{(1-x)^2} &= \frac{1}{1-x} \times \frac{1}{1-x} \\ &= \left(\sum_{i=0}^{\infty} x^i \right) \left(\sum_{i=0}^{\infty} x^i \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^n 1 \times 1 \right) x^n \\ &= \sum_{n=0}^{\infty} (n+1)x^n \\ &= 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots \end{aligned} \quad (3.2)$$

and

$$\begin{aligned}
\frac{1}{(1-x)^3} &= \frac{1}{1-x} \times \frac{1}{1-x}^2 \\
&= \left(\sum_{n=0}^{\infty} x^n \right) \left[\sum_{n=0}^{\infty} (n+1)x^n \right] \\
&= \sum_{n=0}^{\infty} \left[\sum_{i=0}^n 1 \times (n+1-i) \right] x^n \\
&= \sum_{n=0}^{\infty} ((n+1) + n + \dots + 1) x^n \\
&= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n \\
&= 1 + 3x + 6x^2 + 10x^3 + \dots
\end{aligned} \tag{3.3}$$

Before exploring how valuable generating functions are in solving linear homogeneous recurrence relation with constant coefficients, we illustrate how the technique of **partial fraction decomposition**, works to express the quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$ as a sum of proper fractions, where $\deg p(x) < \deg q(x)$.

For example,

$$\frac{6x+1}{(2x-1)(2x+3)} = \frac{1}{2x-1} + \frac{2}{2x+3}$$

partial fraction decomposition rule for $\frac{p(x)}{q(x)}$, where $\deg p(x) < \deg q(x)$

If $q(x)$ has a factor of the form $(ax+b)^m$, then the decomposition contains a sum of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_m}{(ax+b)^m}$$

where A_i is a rational number.

Exercise 3.3.1 Express the following as a sum of partial fractions:

1. $\frac{x}{(1-x)(1-2x)}$
2. $\frac{x}{1-x-x^2}$
3. $\frac{2-9x}{1-6x+9x^2}$

Now we use partial fraction decomposition and generating functions to solve recurrence relations in the following examples.

Example 3.3.2 Use generating function, solve the recurrence relation $b_n = 2b_{n-1} + 1$, where $b_1 = 1$.

Solution

First, notice that the condition $b_1 = 1$ yields $b_0 = 0$. To find the sequence b_n that satisfies the recurrence relation, consider the corresponding generating function

$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots + b_nx^n + \dots$$

Then

$$2xg(x) = 2b_1x^2 + 2b_2x^3 + 2b_3x^4 + \cdots + 2b_{n-1}x^n + \cdots$$

Also

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

Therefore

$$\begin{aligned} g(x) - 2g(x) - \frac{1}{1-x} &= -1 + (b_1 - 1)x + (b_2 - 2b_1 - 1)x^2 + \cdots + (b_n - 2b_{n-1} - 1)x^n + \cdots \\ &= -1 \end{aligned}$$

since $b_1 = 1$ and $b_n = 2b_{n-1} + 1$ for $n \geq 2$. That is,

$$(1 - 2x)g(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

Then

$$\begin{aligned} g(x) &= \frac{x}{(1-2x)(1-x)} \\ &= -\frac{1}{1-x} + \frac{1}{1-2x}, \\ &= -\left(\sum_{n=0}^{\infty} x^n\right) + \left(\sum_{n=0}^{\infty} 2^n x^n\right) \\ &= \sum_{n=0}^{\infty} (2^n - 1)x^n \end{aligned}$$

By partial fraction

But $g(x) = \sum_{n=0}^{\infty} b_n x^n$ so $b_n = 2^n - 1, n \geq 1$ is a solution of the recurrence relation.

Example 3.3.3 Using generating functions, solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, where $a_0 = 2$ and $a_1 = 3$.

Solution

Letting $g(x)$ be the generating function of the sequence in question, Then we have

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

$$6xg(x) = 6a_0x + 6a_1x^2 + 6a_2x^3 + \cdots + 6a_{n-1}x^n + \cdots$$

$$9x^2g(x) = 9a_0x^2 + 9a_1x^3 + 9a_2x^4 + \cdots + 9a_{n-2}x^n + \cdots$$

Then

$$\begin{aligned} g(x) - 6xg(x) + 9x^2g(x) &= a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 9a_0)x^2 + \cdots + (a_n - 6a_{n-1} + 9a_{n-2})x^n + \cdots \\ &= 2 - 9x \end{aligned}$$

Therefore,

$$(1 - 6x + 9x^2)g(x) = 2 - 9x$$

$$\begin{aligned}
g(x) &= \frac{2-9x}{1-6x+9x^2} \\
&= \frac{3}{1-3x} + \frac{3}{(1-3x)^2} \\
&= 3 \left(\sum_{n=0}^{\infty} 3^n x^n \right) - \sum_{n=0}^{\infty} (n+1) 3^n x^n \\
&= \sum_{n=0}^{\infty} (3^{n+1} - (n+1) 3^n) x^n \\
&= \sum_{n=0}^{\infty} (2-n) 3^n x^n
\end{aligned}$$

Thus $a_n = (2-n) 3^n$, $n \geq 0$ is the solution for the recurrence relation.

Example 3.3.4 Solve the recurrence $a_n = -3a_{n-1} + 10a_{n-2}$, $n > 2$, given $a_0 = 1, a_1 = 4$.

Solution

Letting $f(x)$ be the generating function of the sequence in question,

$$\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \\
3xf(x) &= 3a_0 x + 3a_1 x^2 + 3a_2 x^3 + \cdots + 3a_{n-1} x^n + \cdots \\
10x^2 f(x) &= 10a_0 x^2 + 10a_1 x^3 + 10a_2 x^4 + \cdots + 10a_{n-2} x^n + \cdots
\end{aligned}$$

Then

$$\begin{aligned}
f(x) + 3xf(x) - 10x^2 f(x) &= a_0 + (a_1 + 3a_0)x + (a_2 + 3a_1 - 10a_0)x^2 + \cdots + (a_n + 3a_{n-1} - 10a_{n-2})x^n + \cdots \\
&= 1 + 7x
\end{aligned}$$

Therefore

$$(1 + 3x - 10x^2)f(x) = 1 + 7x$$

$$f(x) = \frac{1+7x}{1+3x-10x^2} = \frac{1+7x}{(1+5x)(1-2x)}$$

After applying partial fraction, we arrive at;

$$\begin{aligned}
f(x) &= \frac{1+7x}{(1+5x)(1-2x)} \\
&= \frac{5}{7} \left(\frac{1}{1+5x} \right) (1+7x) + \frac{2}{7} \left(\frac{1}{1-2x} \right) (1+7x) \\
&= \frac{5}{7} (1 + (-5x) + (-5x)^2 + \cdots) (1+7x) + \frac{2}{7} (1 + (2x) + (2x)^2 + \cdots) (1+7x) \\
&= \frac{5}{7} (1 - 5x + 25x^2 + \cdots + (-5)^n x^n + \cdots) (1+7x) + \frac{2}{7} (1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots) (1+7x) \\
&= \frac{5}{7} (1 + 2x - 10x^2 + \cdots + [(-5)^n + 7(-5)^{n-1}] x^n + \cdots) + \frac{2}{7} (1 + 9x + 18x^2 + \cdots + [(2)^n + 7(2)^{n-1}] x^n + \cdots) \\
&= \frac{5}{7} (1 + 2x - 10x^2 + \cdots + 2(-5)^{n-1} x^n + \cdots) + \frac{2}{7} (1 + 9x + 18x^2 + \cdots + 9(2^{n-1}) x^n + \cdots) \\
&= 1 + 4x - 2x^2 + \cdots + \left(-\frac{2}{7} (-5)^n + \frac{9}{7} (2^n) \right) x^n + \cdots
\end{aligned}$$

and hence $a_n = -\frac{2}{7} (-5)^n + \frac{9}{7} (2^n)$ is the desired solution.

Exercise 3.3.2

1. Solve each of the following homogeneous linear recurrences:

$$(a) a_{n+2} - 6a_{n+1} - 7a_n = 0$$

$$(d) a_{n+2} + 10a_{n+1} + 25a_n = 0$$

$$(b) a_{n+3} - 6a_{n+2} + 9a_{n+1} - 4a_n = 0$$

$$(e) a_{n+2} - 4a_{n+1} + 8a_n = 0$$

$$(c) a_{n+3} + 5a_{n+2} + 12a_{n+1} - 18a_n = 0$$

$$(f) a_{n+2} - 6a_{n+1} + 9a_n = 0$$

2. Solve each of the following non-homogeneous linear recurrence relations:

$$(a) a_{n+2} + 4a_{n+1} - 5a_n = 4$$

$$(e) a_{n+2} - 6a_{n+1} + 9a_n = 3^n + 7^n$$

$$(b) a_{n+2} - 2a_{n+1} - 5a_n = \cos n\pi$$

$$(f) a_{n+2} + 4a_{n+1} - 5a_n = n^2 + n + 1$$

$$(c) a_{n+2} - 9a_n = 3^n$$

$$(g) a_{n+2} + 4a_n = n2^n \sin n\pi$$

$$(d) a_{n+2} - 9a_n = n^2 3^n$$

3. Using generating functions, solve the following recurrence relations:

$$(a) a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, n \geq 3, a_0 = 1, a_1 = 3, a_2 = 6$$

$$(b) a_n = a_{n-1} + a_{n-2} - a_{n-3}, n \geq 3, \text{ given } a_0 = 2, a_1 = -1, a_2 = 3$$

$$(c) a_n = 5a_{n-1} - 6a_{n-2}, \text{ given } a_0 = 1, a_1 = 4$$

$$(d) a_n = 4a_{n-1} - 4a_{n-2}, n \geq 2, \text{ given } a_0 = 1, a_1 = 4$$

$$(e) a_n = a_{n-1} + 2, a_1 = 1$$

$$(f) a_n = a_{n-1} + a_{n-2}, a_0 = 2, a_1 = 3$$

Chapter 4

ELEMENTS OF GRAPH THEORY

4.1 Introduction

A graph is conceptually a spatial configuration with a finite set of points and a finite set of lines (possibly curved) joining one point to another (or to itself). Graph theory has its origins in many disciplines. Graphs are natural mathematical models of physical situations in which the points represent either objects or locations and the lines represent connections. Graphs are also used to model sociological and abstract situations in which each line represents a relationship between the entities represented by the points. Applications of graphs are wide-ranging - in areas such as circuit design, communications networks, social interactions, ecology, engineering, operations research, counting, probability, set theory, information theory and sociology.

Graph theory grew out of an interesting physical problem, the celebrated Königsberg Bridge Problem. The outstanding Swiss mathematician Leonhard Euler solved the puzzle in 1736, thus laying the foundation for graph theory and earning his title as the father of graph theory.

The Prussian city of Königsberg (called Kaliningrad during the era of the Soviet Union) lies on the Pregel river. It consists of the two river banks A and C, and the two islands B and D. Seven bridges connect the four land areas of the city.

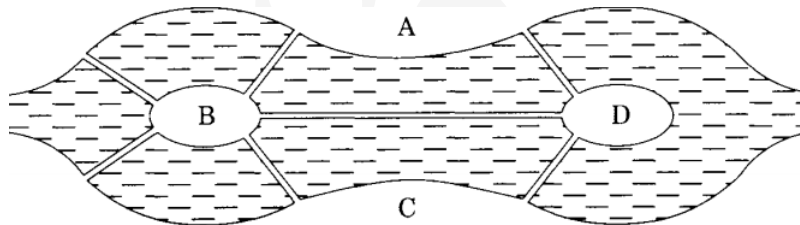


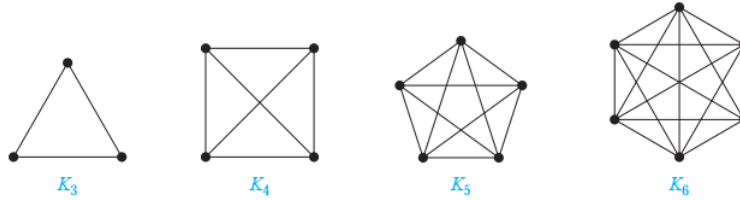
Fig. The City of Königsberg.

Residents of the city used to take evening walks from one part of the city to another. This, naturally, suggested the following question: *Is it possible to walk through the city, traversing each bridge exactly once?* This is the origin of graph theory.

Definition 4.1.1 A graph $G = (V, E)$ consists of V , a nonempty set of vertices (or nodes) and E , a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

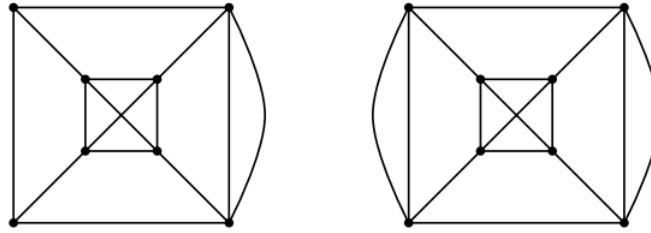
Definition 4.1.2 A *simple graph* $G = (V, E)$ consists of a non-empty set of vertices V and a set of unordered pairs of distinct elements of V called edges E .

Example 4.1.1 The following are examples of simple graphs:



Definition 4.1.3 A **multi graph** $G = (V, E)$ consists of a non-empty set of vertices V and a set of edges E , and a function f from E to $\{\{u, v\} / u, v \in V, u \neq v\}$. The edges e_1 and e_2 are called multiple or parallel edges if $f(e_1) = f(e_2)$.

Example 4.1.2 The following are examples of multi graphs:



Definition 4.1.4 A **pseudo graph** $G = (V, E)$ consists of a set of vertices V , a set of edges E and a function f from E to $\{\{u, v\} / u, v \in V\}$. An edge is a loop if $f(e) = \{u, u\} = \{u\}$ for some $u \in V$.

Example 4.1.3 The following are examples of pseudo graphs: XXXXX

Definition 4.1.5 A graph G_1 is a sub graph of another graph G if and only if the vertex and edge sets of G_1 are, respectively, subsets of the vertex and edge sets of G .

4.2 Matrix representations of graphs

Definition 4.2.1 Two vertices u and v in an undirected graph G are called **adjacent** in G if $\{u, v\}$ is an edge of G . if $e = \{u, v\}$, the edge e is called **incident** with the vertices u and v . the edge e is also connect the vertices u and v . the vertices u and v are called end points of the edges $\{u, v\}$.

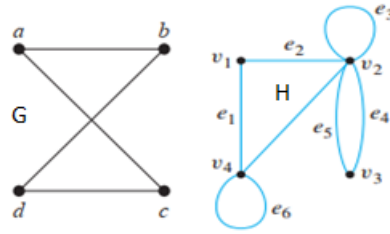
Carrying out graph algorithms using the representation of graphs by lists of edges, or by adjacency lists, can be cumbersome if there are many edges in the graph. To simplify computation, graphs can be represented using matrices. Two types of matrices commonly used to represent graphs will be presented here. One is based on the adjacency of vertices, and the other is based on incidence of vertices and edges.

Adjacency matrix Suppose that $G = (V, E)$ is a simple graph where $|V| = n$ and the vertices of G are listed arbitrary as $v_1, v_2, \dots, v_n$. The adjacency matrix of G denoted by A with respect to this listing of the vertices is the $n \times n$ zero – one matrix with 1 as its $(i, j)^{th}$ entry when v_i and v_j are adjacent, 0 as its $(i, j)^{th}$ entry when they are not adjacent.

In other words, if the adjacency matrix of G is $A = (a_{ij})$ then

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0, & \text{otherwise} \end{cases}$$

Example 4.2.1 Find the adjacency matrix for the graphs G and H shown below:



Solution

We order the vertices of graph G as a, b, c, d . The matrix A representing this graph is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

We order the vertices of graph H as v_1, v_2, v_3, v_4 . The matrix A representing this graph is

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

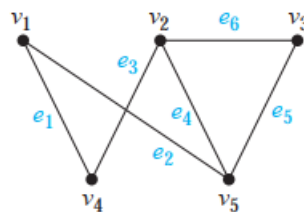
Since the graph H is not simple graph, then the entry of the matrix not zero-one matrix.

Remark 3 the adjacency $n \times n$ square matrix of a simple graph is *symmetric* that is, $a_{ij} = a_{ji}$ since both of these entries are 1 when v_i and v_j are adjacent, and both are zero otherwise.

Incidence matrix The incidence matrix of a graph $G = (V, E)$ is a matrix with rows labeled by vertices $v_1, v_2, \dots, v_n$ and columns labeled by edges $e_1, e_2, \dots, e_m$, so that entry for the incidence matrix with respect to the ordering of v and e is the $n \times m$ matrix $M = (m_{ij})$ where

$$m_{ij} = \begin{cases} 1, & \text{when } e_j \text{ is incident with } v_i \\ 0, & \text{when } e_j \text{ is not incident with } v_i \end{cases}$$

Example 4.2.2 Represent the graph shown below with an incidence matrix.



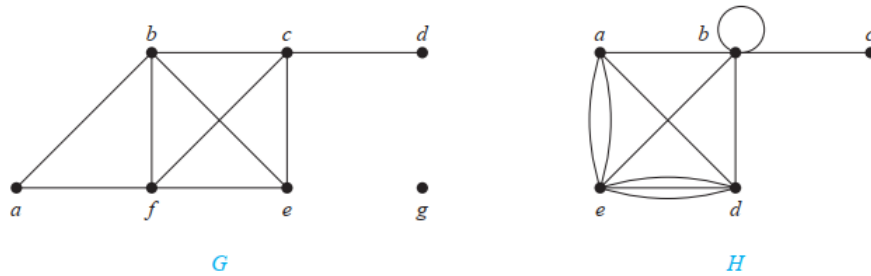
Solution

The incidence matrix M for this graph is a matrix with vertices as rows and edges as columns.

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Definition 4.2.2 The *degree of a vertex v* in an undirected graph G is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

Example 4.2.3 What are the degrees of the vertices in the graphs G and H ?



Solution

From graph G ,

$$\deg(a) = 2, \deg(b) = \deg(c) = \deg(f) = 4, \deg(d) = 1, \deg(e) = 3 \text{ and } \deg(g) = 0.$$

And From graph H ,

$$\deg(a) = 4, \deg(b) = \deg(e) = 6, \deg(c) = 1, \deg(d) = 5$$

Theorem 4.2.1 (Handshaking Theorem) Let $G = (V, E)$ be an undirected graph with m edges. Then

$$\sum_{u \in V} \deg(u) = 2m$$

OR

In every graph, the sum of the degrees of all vertices equals twice the number of edges.

Corollary 3 The total degree of a graph is even.

Example 4.2.4 How many edges are there in a graph with 10 vertices each of degree six?

solution

Because the sum of the degrees of the vertices is $6 \times 10 = 60$, it follows that $2m = 60$ where m is the number of edges. Therefore, $m = 30$.

Theorem 4.2.2 An undirected graph has an even number of vertices of odd degree.

Proof: Let G be a graph with e edges. Let x denote the sum of the degrees of even degree vertices and y the sum of the degrees of odd degree vertices. By Theorem 4.2.1, $x + y = 2e$. Since x is the sum of even integers, x is even, so $y = 2e - x$ is also an even integer. But y is the sum of odd integers, so for y to be even, the number of addends in the sum must be even. In other words, the number of odd degree vertices must be even.

4.3 Isomorphic graphs

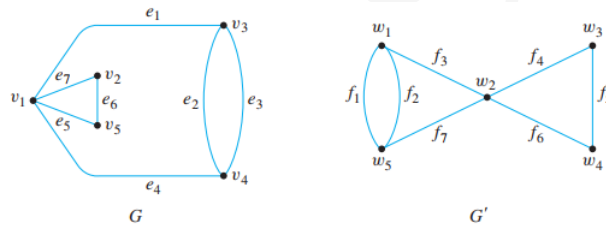
It is important to know when two graphs are essentially the same and when they are essentially different. When we say graphs are “essentially the same,” we mean that they differ only in the way they are labeled or drawn. There should be a one-to-one correspondence between the vertices of the graphs and a one-to-one correspondence between their edges such that corresponding vertices are incident with corresponding edges. The proper word for “essentially the same” is isomorphic.

Definition 4.3.1 Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic graphs if there is a bijection function $f : V(G_1) \rightarrow V(G_2)$ exists such that $\{a, b\}$ is an edge in E_1 if and only if $\{f(a), f(b)\}$ is an edge in E_2 for any two elements a and b in V_1 . Then the function f is called an **isomorphism** between G_1 and G_2 .

In other word, for two simple graphs G_1 and G_2 to be isomorphic, the following conditions must be satisfied;

- $|V_1| = |V_2|$ and $|E_1| = |E_2|$
- A bijection function $f : V_1 \rightarrow V_2$ should preserve the adjacency relationship, if $\{a, b\}$ is an edge in E_1 then $\{f(a), f(b)\}$ must be an edge in E_2 and vice versa. Consequently, the corresponding vertices in G_1 and G_2 will have the same degree.

Example 4.3.1 Are the graphs G and G' isomorphic?



Solution: To check these two graphs are isomorphic, we have to check the following conditions:

$$|V_1| = |V_2| = 5, \quad |E_1| = |E_2| = 7$$

and we must find a function $f : V_1 \rightarrow V_2$ as $f(v_1) = w_2, f(v_2) = w_3, f(v_3) = w_1, f(v_4) = w_5$ and $f(v_5) = w_4$. Then f is bijection function.

Does f preserve the adjacency relationship?

$\{v_1, v_2\} = e_7$ is an edge in E_1 , then $\{f(v_1), f(v_2)\} = \{w_2, w_3\} = f_4$ is an edge in E_2 .

$\{v_1, v_3\} = e_1$ is an edge in E_1 , then $\{f(v_1), f(v_3)\} = \{w_2, w_1\} = f_3$ is an edge in E_2 .

$\{v_1, v_4\} = e_4$ is an edge in E_1 , then $\{f(v_1), f(v_4)\} = \{w_2, w_5\} = f_7$ is an edge in E_2 .

$\{v_1, v_5\} = e_5$ is an edge in E_1 , then $\{f(v_1), f(v_5)\} = \{w_2, w_4\} = f_6$ is an edge in E_2 .

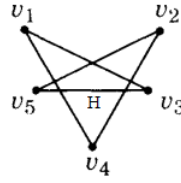
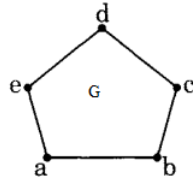
$\{v_2, v_5\} = e_6$ is an edge in E_1 , then $\{f(v_2), f(v_5)\} = \{w_3, w_4\} = f_5$ is an edge in E_2 .

$\{v_3, v_4\} = e_2$ is an edge in E_1 , then $\{f(v_3), f(v_4)\} = \{w_1, w_5\} = f_1$ is an edge in E_2 .

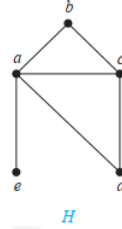
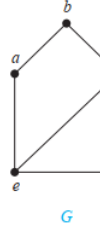
$\{v_3, v_4\} = e_3$ is an edge in E_1 , then $\{f(v_3), f(v_4)\} = \{w_1, w_5\} = f_2$ is an edge in E_2 . Therefore f preserves the adjacency relationship. Hence G and G' are isomorphic graphs.

Exercise 4.3.1

1. Show that weather the following graphs are isomorphic or not?



2. Show that whether the following graphs are isomorphic or not?



proposition 1 If G_1 and G_2 are isomorphic graphs, then G_1 , and G_2 have the

- same number of vertices,
- same number of edges, and
- same degree sequences.

Theorem 4.3.1 Two graphs are isomorphic if and only if their vertices can be labeled in such a way that the corresponding adjacency matrices are equal.

4.4 Path, Cycles, Circuit and Connectivity of a Graph

Introduction

Many problems can be modeled with paths formed by traveling along the edges of graphs. For instance, the problem of determining whether a message can be sent between two computers using intermediate links can be studied with a graph model. Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on can be solved using models that involve paths in graphs.

Definition 4.4.1 Let v_0 and v_n be two vertices of a multi graph G , a **path (Walk)** of length n from v_0 to v_n is an alternating sequence of vertices v_i and edges e_i of the form

$$v_0 - e_1 - v_1 - e_2 - v_2 - \cdots - e_n - v_n$$

Where each edge e_i is incident with the vertices v_{i-1} and v_i , $1 \leq i \leq n$.

If the graph is simple graph, the path is unique and is denoted by just listing the vertices along the path:

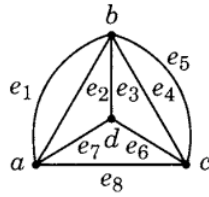
$$v_0 - v_1 - v_2 - \cdots - v_n$$

where v_0 and v_n are the end points of the path.

A **simple path** is a path from v_0 and v_n contains no repeated vertices with one possible exception; its end points could be the same.

A **cycle** is a simple closed path with endpoints $v_0 = v_n$. A closed path with no repeated edges is called **circuit**. A circuit is called a cycle if no vertex is repeated except the initial and terminal vertices.

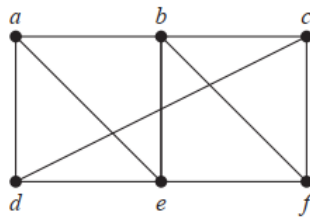
Example 4.4.1 Consider the graph below:



In this graph, the sequence $a - e_1 - b - e_4 - c - e_5 - b - e_3 - d$ is a path of length 4 from a to d and also the sequence $a - e_1 - b - e_5 - c - e_4 - b - e_3 - d$ then the path from a to d is not unique.

The path $a - e_7 - d - e_3 - b$ is simple path since no vertex is repeated. The sequence $a - e_1 - b - e_3 - d - e_6 - c - e_8 - a$ is simple path (cycle or circuit) and $a - e_8 - c - e_6 - d - e_3 - b - e_4 - c - e_5 - b - e_1 - a$ is an example of circuit but not simple path.

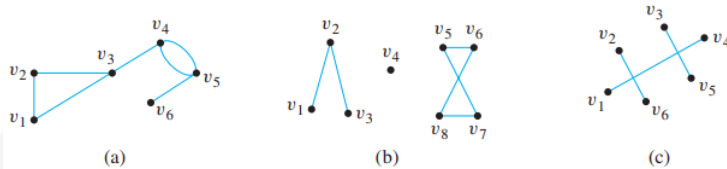
Example 4.4.2 Let's consider the following graph:



In this simple graph, $a - d - c - f - e$ is a simple path of length 4. Since $\{a, d\}$, $\{d, c\}$, $\{c, f\}$ and $\{f, e\}$ are all edges of the graph. $d - e - c - a$ is not a path since $\{e, c\}$ is not an edge of the graph. The path $a - b - e - d - a - b$ of length 5 is not simple path since it contains the edge $\{a, b\}$ twice.

Definition 4.4.2 A graph G is **connected** if there is a path between every pair of distinct vertices of the graph (OR a graph is connected if there is a path between any two of its vertices); otherwise, it is disconnected.

Example 4.4.3 Which of the following graphs are connected?



Solution: The graph represented in (a) is connected, whereas those of (b) and (c) are not.

Theorem 4.4.1 There is a simple path between every pair of distinct vertices of a connected undirected graph.

Theorem 4.4.2 The length of a simple path between any two distinct vertices of a connected graph with n vertices is at most $n - 1$.

Definition 4.4.3 A graph H is a connected component of a graph G if and only if,

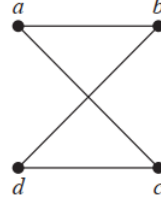
- H is sub-graph of G ;
- H is connected; and
- no connected sub-graph of G has H as a sub-graph and contains vertices or edges that are not in H .

Counting Paths Between Vertices

The number of paths between two vertices in a graph can be determined using its adjacency matrix.

Theorem 4.4.3 Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \dots, v_n$ of the vertices of the undirected graph (with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j , where r is a positive integer, equals the $(i, j)^{\text{th}}$ entry of A^r .

Example 4.4.4 How many paths of length four are there from a to d in the simple graph below:



Solution: The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Hence, the number of paths of length four from a to d is the $(1, 4)^{\text{th}}$ entry of A^4 . Because

$$A^4 = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}$$

there are exactly eight paths of length four from a to d .

We conclude this section with an example illustrating how useful graphs and their paths can be in solving familiar and interesting puzzles.

Example 4.4.5 (The cabbage-goat-wolf puzzle) A farmer with a rowboat needs to transport a cabbage, a goat, and a wolf across a river. The rowboat has just enough room for him and either the cabbage, the goat, or the wolf. Since the wolf can eat the goat, they cannot be left alone in the absence of the farmer. Likewise, the goat and the cabbage also cannot be left alone. How can he transfer them across the river?

Solution: We shall represent the progress of a solution in a graph and update it at every step until a solution emerges. An edge indicates a transfer across the river.

Initially the cabbage (C) and the goat (G) or the goat and the wolf (W) cannot be left unattended by the farmer (F). The cabbage and the wolf, however, can be left alone. So the first step is to transfer the goat across the river. We represent this by an edge connecting the vertices "start" (S) and FG, meaning both the farmer and the goat must go across the river (see Fig.1a). Now the man must return to the original side (see Fig.1b) and has then two choices: transport the cabbage or the wolf to the other side (see Fig.1c).

S — FG

Fig. 1a.

S — FG — F

Fig. 1b.

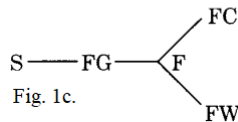
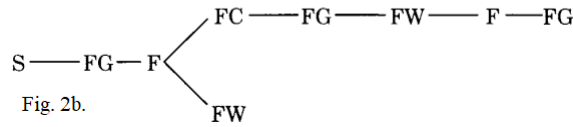
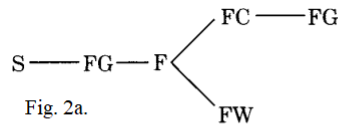
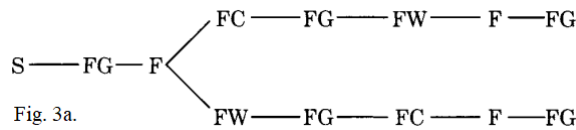


Fig. 1c.

Case 1 Suppose the farmer moves the cabbage across the river. Then he cannot leave the cabbage and the goat on the same side, so he must return with the goat (see Fig.2a). Continuing like this yields the graph in Fig.2b.



Case 2 Suppose the man transfers the wolf across the river. Continuing as in case 1 produces the graph in Fig.3a.

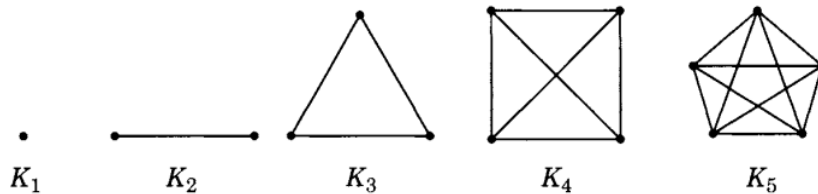


It follows from this graph that the puzzle has exactly two solutions, given by the two paths beginning at S.

4.5 Complete, Regular and Bipartite Graphs

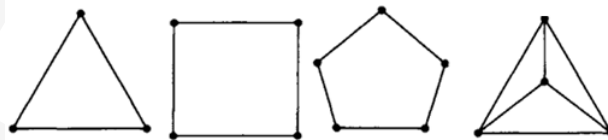
Complete Graph A simple graph that contains every possible edge between all the vertices is called a **complete graph**. A complete graph with n vertices is denoted as K_n .

Example 4.5.1 The following graphs are an examples of complete graphs:



Regular Graph A regular graph is a simple graph whose vertices have all the same degree.

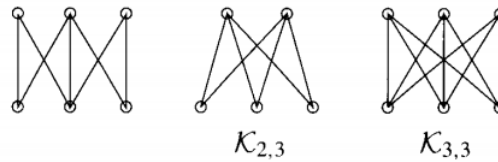
Example 4.5.2 The following graphs are an examples of regular graphs:



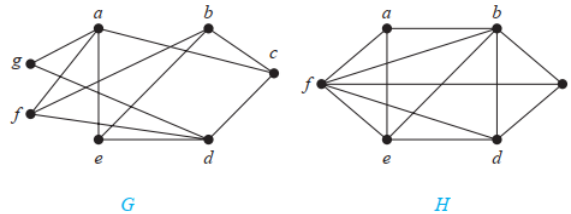
Bipartite Graph A simple graph G is called **bipartite graph**, if it's vertex set V can be partitioned in to two disjoint nonempty sets X and Y such that every edge in the graph connects a vertex in X and a vertex in Y (so that no edge in G connects either two vertices in X or two vertices in Y).

A bipartite simple graph G is called **complete bipartite graph** if each vertex in X is connected to each vertex in Y . if $|X| = m$ and $|Y| = n$, then the corresponding complete bipartite graph is denoted by $K_{m,n}$.

Example 4.5.3 The following graphs are an examples of bipartite graphs:



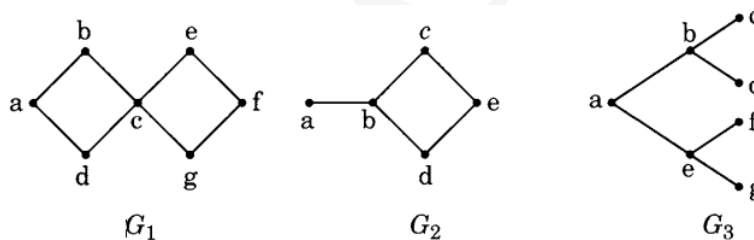
Exercise 4.5.1 Are the graphs G and H bipartite graphs?



4.6 Eulerian and Hamiltonian Graphs

Definition 4.6.1 A path in a connected graph is an **Eulerian path** if it contains every edge exactly once. A circuit in a connected graph is an **Eulerian circuit** if it contains every edge of the graph. A connected graph with an Eulerian circuit is an **Eulerian graph**.

Example 4.6.1 Determine whether the given graphs have an Eulerian circuit. Construct such a circuit when one exists. If no Eulerian circuit exists, determine whether the graphs have an Eulerian path and construct such a path if one exists.

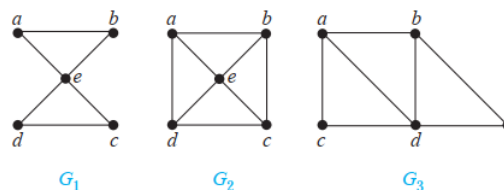


Solution: G_1 is Eulerian graph since it contains an Eulerian circuit, for example, $a-b-c-e-f-g-c-d-a$; in fact, it contains several Eulerian circuits. (Can you find another one?)

G_2 has an Eulerian path, namely, $a-b-c-e-d-b$, but no Eulerian circuits and G_3 has no Eulerian paths or circuits.

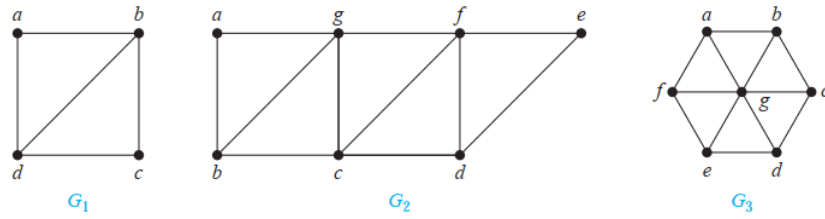
Theorem 4.6.1 A connected graph G is Eulerian graph if and only if every vertex of G has even degree.

Exercise 4.6.1 Which of the undirected graphs in below have an Eulerian circuit? Of those that do not, which have an Eulerian path?



Theorem 4.6.2 A connected graph contains an Eulerian path, but not an Eulerian circuit, if and only if it has exactly two vertices of odd degree.

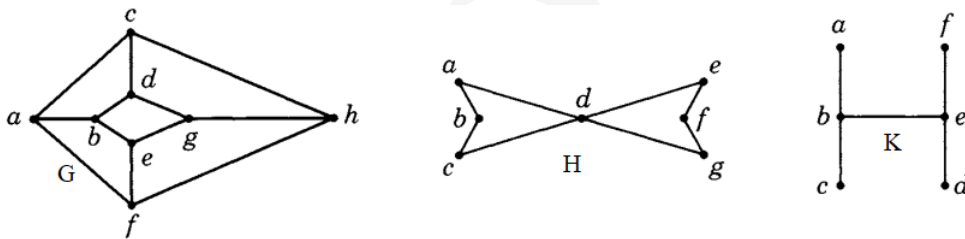
Example 4.6.2 Determine if each of the following graph has an Eulerian path. If so, find it.



Solution: Solution: G_1 contains exactly two vertices of odd degree, namely, b and d . Hence, it has an Eulerian path that must have b and d as its endpoints. One such Eulerian path is $d - a - b - c - d - b$. Similarly, G_2 has exactly two vertices of odd degree, namely, b and d . So it has an Eulerian path that must have b and d as endpoints. One such Euler path is $b - a - g - f - e - d - c - g - b - c - f - d$. G_3 has no Eulerian path because it has six vertices of odd degree.

Definition 4.6.2 A simple path in a connected graph is **Hamiltonian path** if it contains every vertex. A cycle in a connected graph that contains every vertex is **Hamiltonian cycle**. A connected graph that contains a Hamiltonian cycle is a **Hamiltonian graph**.

Example 4.6.3 Are the following graphs Hamiltonian? If a graph is not Hamiltonian, does it contain a Hamiltonian path?



Solution: Graph G contains a Hamiltonian cycle, for example, $a - c - d - g - h - f - e - b - a$, so it is Hamiltonian graph.

In H , suppose we start at a , b , or c . Then we have to pass through d to visit vertices e , f , and g . So, to return to the home vertex, we have to pass through d again. Thus H contains no Hamiltonian cycles that begin at a , b , or c . Suppose we start at d . Then again after visiting a , b , and c , we have to pass through d to visit e , f , or g before returning to d . Consequently, H contains no Hamiltonian cycles that begin at d either. Suppose we start at e , f , or g . This also yields no Hamiltonian cycles. Thus H is not Hamiltonian graph. But, it does contain a Hamiltonian path, $a - b - c - d - e - f - g$.

In K , to visit a , c , d , or f , we must enter and leave both b and c at least twice, so it is not Hamiltonian. By a similar argument, it has no Hamiltonian paths.

Theorem 4.6.3 (DIRAC'S THEOREM) If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Theorem 4.6.4 (ORE'S THEOREM) If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of non adjacent vertices u and v in G , then G has a Hamilton circuit.

4.6.1 Applications of Hamilton Circuits

Hamilton paths and circuits can be used to solve practical problems. For example, many applications ask for a path or circuit that visits each road intersection in a city, each place pipelines intersect in a utility grid, or each node in a communications network exactly once. Finding a Hamilton path or circuit in the appropriate graph model can solve such problems. The famous **traveling salesperson problem** or **TSP** asks for the shortest route a traveling salesperson should take to visit a set of cities. This problem reduces to finding a Hamilton circuit in a complete graph such that the total weight of its edges is as small as possible. We will return to this question in chapter 6.

4.7 Trees

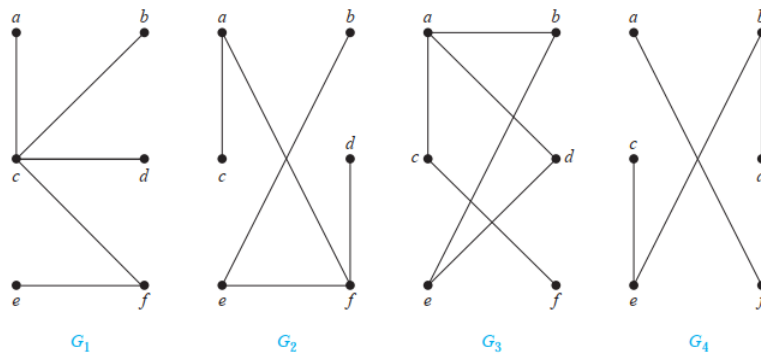
Introduction

Trees are the most important class of graphs and they make fine modeling tools. In 1847, the German physicist Gustav Robert Kirchoff used them to solve systems of linear equations for electrical networks. Ten years later, Arthur Cayley studied the isomers of saturated hydrocarbons C_nH_{2n+2} with them. Today trees are widely used in mathematics and computer science, as well as in linguistics and the social sciences.

Trees are particularly useful in computer science, where they are employed in a wide range of algorithms. For instance, trees are used to construct efficient algorithms for locating items in a list. They can be used in algorithms, such as Huffman coding, that construct efficient codes saving costs in data transmission and storage. Trees can be used to study games such as checkers and chess and can help determine winning strategies for playing these games. Trees can be used to model procedures carried out using a sequence of decisions. Constructing these models can help determine the computational complexity of algorithms based on a sequence of decisions, such as sorting algorithms.

Definition 4.7.1 A **tree** is a connected undirected graph which contains no simple circuits. A graph is called a **forest** if, and only if, it is circuit-free and not connected.

Example 4.7.1 Which of the graphs shown below are trees?



Solution: G_1 and G_2 are trees, because both are connected graphs with no simple circuits. G_3 is not a tree because e, b, a, d, e is a simple circuit in this graph. Finally, G_4 is not a tree because it is not connected. but G_4 is forest.

Theorem 4.7.1 An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Lemma 1 Any tree that has more than one vertex has at least one vertex of degree 1.

proposition 2 A graph T with n vertices is called a tree if

1. T is connected and is circuitless (or)

2. T is connected and has $n - 1$ edges (or)
3. T is circuitless and has $n - 1$ edges (or)
4. There is exactly one path between every pair of vertices in T .

Theorem 4.7.2 A connected graph with n vertices is a tree if and only if it has $n - 1$ edges.

Proof: First we prove by mathematical induction, that a tree with n vertices has $n - 1$ edges.

Let $P(n)$: A tree T with n vertices has $n - 1$ edges.

Basis step Suppose T contains one vertex. Since T is acyclic, it is loop free; consequently, T contains no edges and $P(1)$ is true.

Induction step Suppose the result is true for every tree with k or fewer vertices. Let T be a tree with $k + 1$ vertices and e an edge in T . Deleting e from T yields two connected disjoint graphs, T_1 and T_2 . Each is a tree. Suppose T_1 has p vertices; then T_2 has $q = k + 1 - p$ vertices. By the inductive hypothesis, T_1 contains $p - 1$ edges and T_2 contains $q - 1$ edges; so T contains $(p - 1) + (q - 1) + 1 = (p - 1) + (k - p) + 1 = k$ edges. Thus by mathematical induction, $P(n)$ is true for every $n \geq 1$.

Conversely, let G be a connected graph with n vertices and $n - 1$ edges. If it were not a tree, it would have a cycle. Remove an edge from this cycle. The resulting graph is still connected. If it is not acyclic, remove an edge from a cycle. Continue this procedure to get an acyclic graph H . Thus H is connected and acyclic; it must be a tree with n vertices. So, by the first part, H has $n - 1$ edges and hence G has more than $n - 1$ edges, which is a contradiction. Thus G is connected and acyclic, and hence a tree.

4.7.1 Rooted Tree

In many applications of trees, a particular vertex of a tree is designated as the **root**. Once we specify a root, we can assign a direction to each edge as follows. Because there is a unique path from the root to each vertex of the graph, we direct each edge away from the root. Thus, a tree together with its root produces an undirected graph called a **rooted tree**.

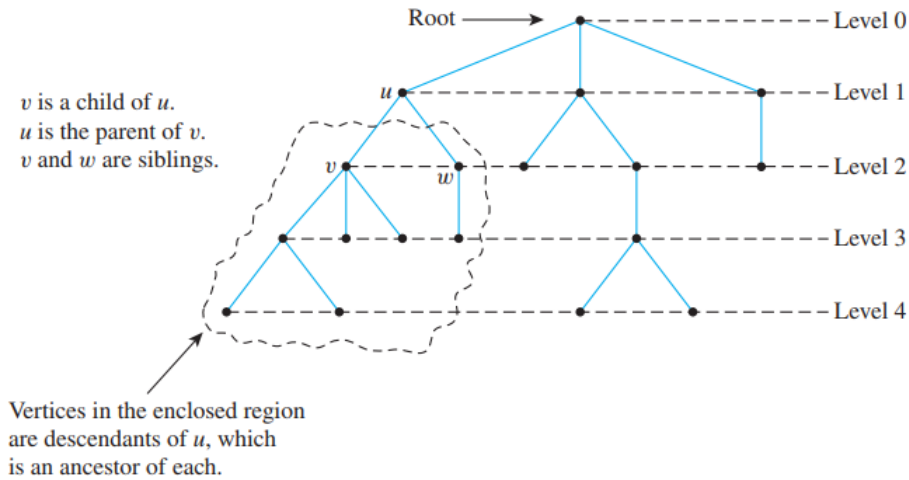
Definition 4.7.2 A **rooted tree** is a tree in which there is one vertex that is distinguished from the others and is called the **root**. The **level** of a vertex is the number of edges along the unique path between it and the root. The **height** of a rooted tree is the maximum level of any vertex of the tree.

Remark 4

- i) Let v and y are vertices of tree T , if the level of y is greater than the level of v , then we say y is below v . If y below v and there is an edge from v to y , then y is a son (child) of v and v is a parent of y .
- ii) If $p = (v_0, v_1, v_2, \dots, v_n)$ is a path from $v = v_0$ to $y = v_n$ and for each $i = 0$ to $n - 1$, v_{n+1} is the child of v_i then y is called a **descendant** of v .
- iii) The vertex v together with all its descendants is called the sub-tree of T rooted at v .
- iv) A vertex with no child is called a **leaf**.
- v) All vertices including the root that are not leaves are called **interior vertices**.
- vi) A rooted tree is called an **m -ary tree** if every interior vertex has m or fewer children. The tree is called a **complete (full) m -ary tree** if every interior vertex has exactly m -children. m -ary tree with $m = 2$ is called a **binary tree**.
- vii) The **ancestors** of a vertex other than the root are the vertices in the path from the root to the vertex excluding the vertex it self and including the root.

viii) vertices with the same parent are called *siblings*.

Example 4.7.2 Let's see the following rooted tree :



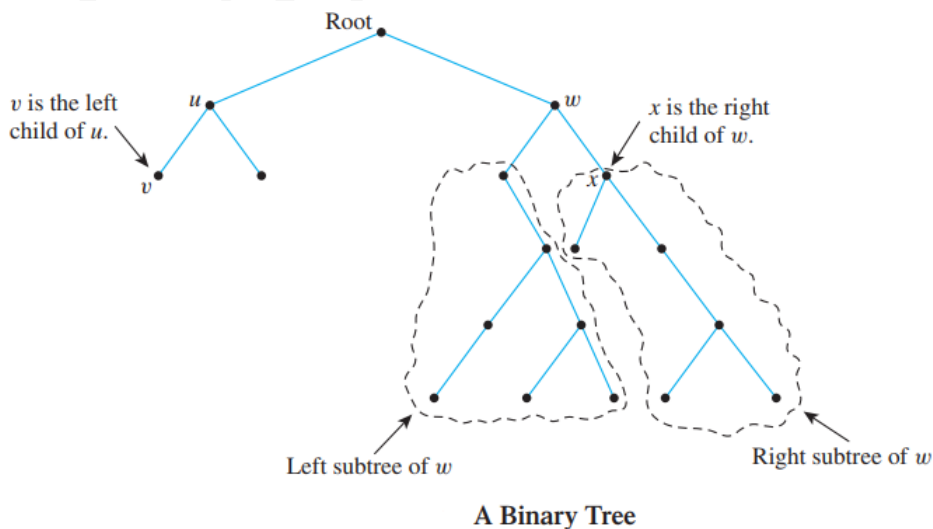
the height of this rooted tree is 4, it has 9 interior vertices and has 9 leaves.

Definition 4.7.3 A rooted tree of height h is said to be balanced tree if each leaf is at level h or $h - 1$.

Definition 4.7.4 A Binary tree is a tree in which there is exactly one vertex of degree two and each of the remaining vertices of degree one or three. The vertex of degree two serves as a root.

An **ordered rooted tree** is a rooted tree where the children of each internal vertex are ordered. Ordered rooted trees are drawn so that the children of each internal vertex are shown in order from left to right. Note that a representation of a rooted tree in the conventional way determines an ordering for its edges. We will use such orderings of edges in drawings without explicitly mentioning that we are considering a rooted tree to be ordered. In an ordered binary tree (usually called just a binary tree), if an internal vertex has two children, the first child is called the **left child** and the second child is called the **right child**. The tree rooted at the left child of a vertex is called the **left sub-tree** of this vertex, and the tree rooted at the right child of a vertex is called the **right sub-tree** of the vertex.

These terms are illustrated in the following Tree.



4.7.2 Tree Traversal

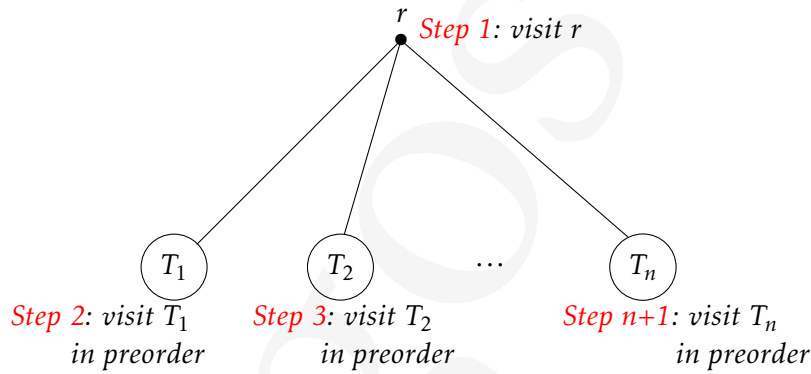
Introduction

Ordered rooted trees are often used to store information. We need procedures for visiting each vertex of an ordered rooted tree to access data. We will describe several important algorithms for visiting all the vertices of an ordered rooted tree. Ordered rooted trees can also be used to represent various types of expressions, such as arithmetic expressions involving numbers, variables, and operations. The different listings of the vertices of ordered rooted trees used to represent expressions are useful in the evaluation of these expressions.

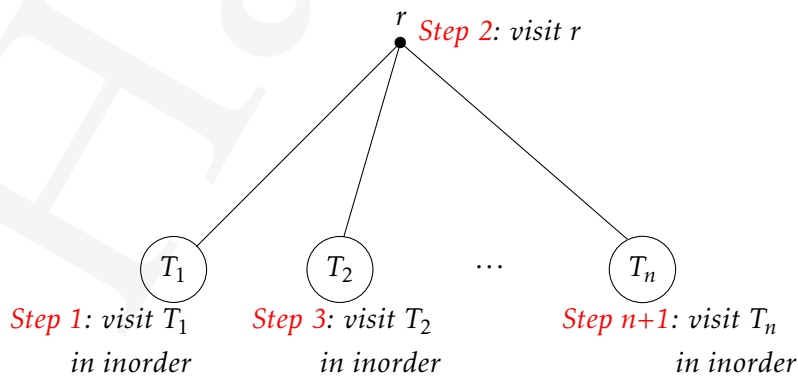
Traversal Algorithms

An algorithm is a finite step by step list of well-defined instructions for solving a particular problem. Procedures for systematically visiting every vertex of an ordered rooted tree are called traversal algorithms. We will describe three of the most commonly used such algorithms, **preorder traversal**, **inorder traversal**, and **postorder traversal**. Each of these algorithms can be defined recursively.

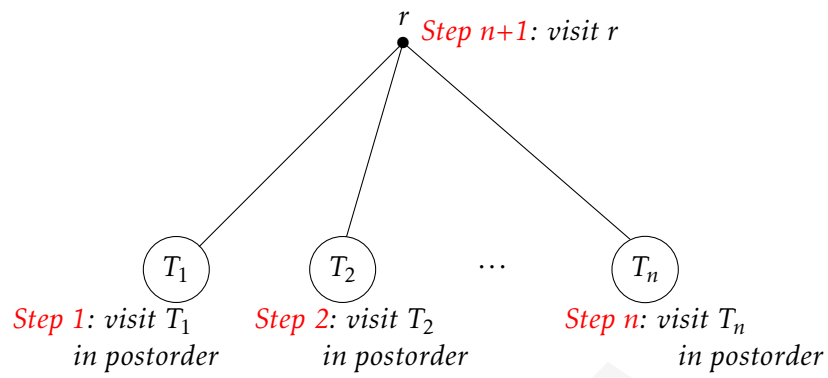
Definition 4.7.5 Let T be an ordered rooted tree with root r . If T consists only of r , then r is the preorder traversal of T . Otherwise, suppose that $T_1, T_2, \dots, T_n$ are the sub-trees at r from left to right in T . The **preorder traversal** begins by visiting r . It continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.



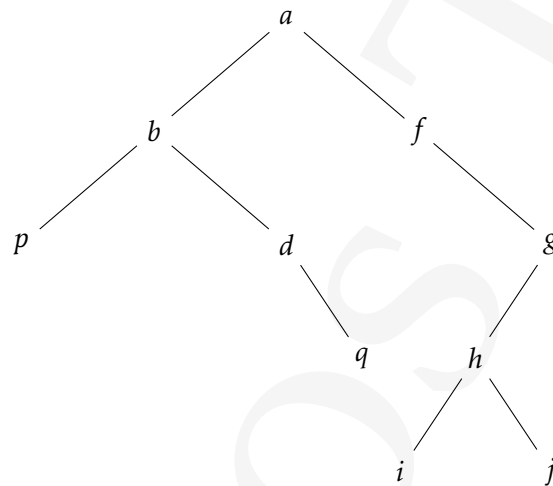
Definition 4.7.6 Let T be an ordered rooted tree with root r . If T consists only of r , then r is the inorder traversal of T . Otherwise, suppose that $T_1, T_2, \dots, T_n$ are the sub-trees at r from left to right. The **inorder traversal** begins by traversing T_1 in inorder, then visiting r . It continues by traversing T_2 in inorder, then T_3 in inorder, $\dots$, and finally T_n in inorder.



Definition 4.7.7 Let T be an ordered rooted tree with root r . If T consists only of r , then r is the postorder traversal of T . Otherwise, suppose that $T_1, T_2, \dots, T_n$ are the sub-trees at r from left to right. The **postorder traversal** begins by traversing T_1 in postorder, then T_2 in postorder, $\dots$, then T_n in postorder, and ends by visiting r .



Example 4.7.3 Find the preorder, inorder, postorder traversal of the following ordered rooted tree:



Solution

Preorder traversal: *abpdqfghij.*

Inorder traversal: *pbdqafihjg.*

Postorder traversal: *pgdbijhgf a.*

Chapter 5

DIRECTED GRAPHS (DIGRAPHS)

Chapter 6

WEIGHTED GRAPHS AND THEIR APPLICATIONS